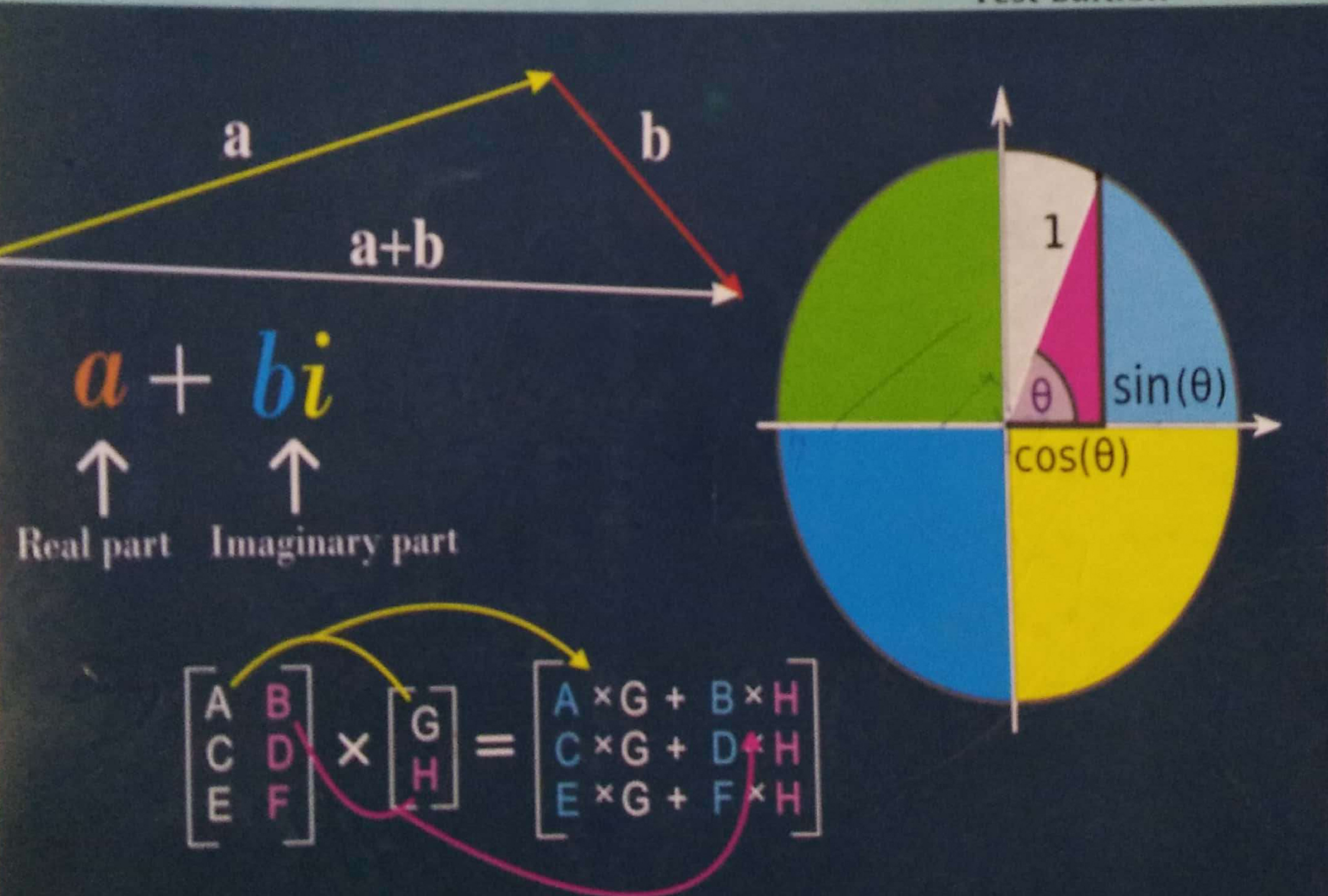
A TEXTBOOK OF

MATHEMATICS

FOR GRADE XI

Test Edition



Khyber Pakhtunkhwa Textbook Board,

Peshawar

A Textbook of

MATHEMATICS

For Grade XI

Khyber Pakhtunkhwa Textbook Board,
Peshawar

CONTENTS

UNIT 1

COMPLEX NUMBERS

$a + bi$

↑ Real part ↑ Imaginary part

After reading this unit, the students will be able to:

STUDENTS LEARNING OUTCOMES

- Recall complex number z represented by an expression of the form $z=a+ib$ or of the form (a,b) where a and b are real numbers and $i=\sqrt{-1}$
- Recognize a as real part of z and b as imaginary part of z
- Know the condition for equality of complex numbers
- Carry out basic operations on complex numbers
- Define $\overline{z} = a - ib$ as the complex conjugate of $z=a+ib$
- Define $|z|=\sqrt{a^2+b^2}$ as the absolute value or modulus of a complex number $z=a+ib$
- Describe algebraic properties of complex numbers (e.g. commutative, associative and distributive) with respect to '+' and '×'
- Know additive identity and multiplicative identity for the set of complex numbers
- Find additive inverse and multiplicative inverse of a complex number z
- Demonstrate the following properties
  - $|z|=|-z|=|\overline{z}|=|-\overline{z}|$
  - $\overline{\overline{z}}=z$, $z\overline{z}=|z|^2$, $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$,
  - $\overline{z_1 z_2}=\overline{z_1}\,\overline{z_2}$, $\overline{\left(\dfrac{z_1}{z_2}\right)}=\dfrac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$.
- Find real and imaginary parts of the following type of complex numbers
  - $(x+iy)^n$,
  - $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n$, $x_2+iy_2 \neq 0$, where n = ±1, and ±2
- Solve the simultaneous linear equations with complex coefficients. For example,
  $$\begin{cases} 5z-(3+i)w=7-i, \\ (2-i)z+2iw=-1+i \end{cases}$$
- Write the polynomial p(z) as a product of linear factors. For example,
  $z^2+a^2=(z+ia)(z-ia)$, $z^3-3z^2+z+5=(z+1)(z-2-i)(z-2+i)$
- Solve quadratic equation of the form $pz^2+qz+r=0$ by completing squares, where p, q, r are real numbers and z a complex number. For example:
  Solve $z^2-2z+5=0 \Rightarrow (z-1-2i)(z-1+2i)=0 \Rightarrow z=1+2i, 1-2i$

1.1 Introduction

In our previous class we learnt that besides the real numbers, there are other numbers called complex numbers. Such numbers play a very important role in mathematics and other branches of science. The use of complex numbers is indispensable in Physics, Aeronautical and Electrical Engineering especially in the analysis of Electric circuits.

1.1.1 Complex Numbers

In 1832, Gauss (1777-1855), a German mathematician gave the concept of **complex numbers** as numbers of the form $a+bi$, where a and b are real numbers. The number a is called the **real part** of $a+bi$ and the number b is called the **imaginary part** of $a+bi$.

For example, the complex number $-3+2i$ has the real part $a = -3$ and the imaginary part $b = 2$.

In $a + bi$, if $b = 0$, then $a + bi = a + 0i = a$ is a real number. Thus every real number a can be written as a complex number by choosing $b = 0$. If $a = 0$ and $b \neq 0$, then $a + bi = 0 + bi = bi$ is called a **pure imaginary number.**

For example, $\frac{1}{4}i$ and $-i$ are pure imaginary numbers. Usually, the complex number $a + bi$ is denoted by $z = a + bi$

Accordingly, $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i, \ldots$

The set of all complex numbers is denoted by $\mathbb{C}$, that is $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.

Complex numbers may also be defined as ordered pairs of real numbers. Thus a **complex number** z is an ordered pair (a, b) of real numbers a and b, written as $z = (a, b)$. The first component a is called the **real part** of z and the second component b is called the **imaginary part** of z denoted by $Re(z)$ and $Im(z)$ respectively i.e. $Re(z) = a$ and $Im(z) = b$.

The ordered pair $(0,1)$ is called the **imaginary unit** and is denoted by $i = (0,1)$.

The set of all ordered pairs of real numbers is the set of complex numbers denoted by $\mathbb{C}$, that is $\mathbb{C} = \{(a, b) : a, b \text{ are real numbers}\}$

$$= \mathbb{R} \times \mathbb{R} \text{ where } \mathbb{R} \text{ is the set of real numbers.}$$

Since $i = \sqrt{-1}$, a simple consequence of the definition of i is that all powers of i may be expressed in terms of ± 1 and $\pm i$.

For example, $i^1 = i$, $i^2 = -1$, $i^3 = i^2 i = -i$, $i^4 = (i^2)^2 = 1$ and if we continue in this way to obtain higher powers of i, we obtain the values $1, i, -1$ or $-i$. Similarly, for negative powers, we have

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

$$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1$$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{i^2 \cdot i} = \frac{1}{-i} = \frac{i}{-i \cdot i} = i$$

$$i^{-4} = \frac{1}{i^4} = \frac{1}{(i^2)^2} = \frac{1}{(-1)^2} = 1$$

> **Note**
>
> In Example 1(c) we see that 0 can be expressed as a sum of a real and an imaginary number and hence is a complex number. Such a complex number whose real and imaginary parts are zero, is called **zero complex number.**
>
> Similarly in Example 1(d), 1 can be expressed as a complex number with real part 1 and imaginary part 0. The complex number 1 is called the **unit complex number.**

Example 1: Write the following complex numbers in ordered pair form.

(a) 6 (b) $5i$ (c) 0 (d) 1 (e) $3-\sqrt{-9}$

Solution:

(a) $6 = 6 + 0i = (6,0)$ (b) $5i = 0 + 5i = (0,5)$

(c) $0 = 0 + 0i = (0,0)$ (d) $1 = 1 + 0i = (1,0)$

(e) $3-\sqrt{-9} = 3 - i\sqrt{9} = 3 - 3i = (3, -3)$

Example 2: Find the value of

$$\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} - 1$$

Solution: Given expression

$$= \frac{i^{10}(i^{582} + i^{580} + i^{578} + i^{576} + i^{574})}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} - 1 = i^{10} - 1$$

$$= (i^2)^5 - 1 = (-1)^5 - 1 = -1 - 1 = -2$$

1.1.2 Equality of Complex Numbers

Two complex numbers are said to be equal if and only if their corresponding real parts and imaginary parts are equal. i.e. $a + ib = c + id \Leftrightarrow a = c$ and $b = d$

i.e. $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

Illustration: If $x+iy = 3-4i$, then $x = 3$ and $y = -4$

1.1.3 Conjugate of a complex number

The conjugate of a complex number $z = x + iy$ is denoted by $\overline{z}$, and is defined as $z = x - \overline{iy}$

Illustration: (i) Let $z = 5 - 3i$, then $\overline{z} = 5 + 3i$

(ii) Let $z = 2 = 2+0i$, then $\overline{z} = 2 - 0i = 2$

(iii) Let $z = 3i = 0 + 3i$, then $\overline{z} = 0 - 3i = -3i$

1.1.4 Basic algebraic operation on complex numbers

(i) Addition

For two complex numbers, $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, their sum is defined as:

$$z = z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

Illustration: If $z_1 = 4 + 5i$ and $z_2 = 2 - 3i$, then $z_1 + z_2 = (4+2) + (5-3)i = 6 + 2i$

Example 3: Add the complex numbers $z_1 = 3 + 4i$ and $z_2 = 2 - 7i$

Solution: $z_1 + z_2 = (3+4i) + (2-7i)$
$= (3+2) + (4-7)i = 5 - 3i$

(ii) Subtraction

For two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, the subtraction of z_2 from z_1 is defined as:

$$z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$$

Illustration: If $z_1 = 1 - i$ and $z_2 = 5 + 2i$, then

$z_1 - z_2 = (1-i) - (5+2i) = (1-i) + (-5-2i)$
$= (1-5) + i(-1-2) = -4 - 3i$

Remember

For any two real numbers a and b, $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is true only when at least one of a and b is either zero or positive.
If both a and b are positive real numbers, then the calculation

$\sqrt{-a}\sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{ab}$

is wrong. The correct calculation is

$\sqrt{-a}\sqrt{-b} = (\sqrt{-1}\sqrt{a})(\sqrt{-1}\sqrt{b})$

$= (i\sqrt{a})(i\sqrt{b})$

$= i^2(\sqrt{a}\sqrt{b}) = (-1)(\sqrt{ab}) = -\sqrt{ab}$

Thus, the calculation

$\sqrt{-2}\sqrt{-3} = \sqrt{(-2)(-3)} = \sqrt{6}$

is wrong. The correct result is

$\sqrt{-2}\sqrt{-3} = (i\sqrt{2})(i\sqrt{3})$

$= i^2(\sqrt{2}\sqrt{3}) = -\sqrt{6}$

Example 4: Let $z_1 = 2 + 4i$ and $z_2 = 1 - 3i$. Compute $z_1 - 3z_2$

Solution: Putting values of z_1 and z_2 in the given expression,

$$z_1 - 3z_2 = (2 + 4i) - 3(1 - 3i)$$
$$= 2 + 4i - 3 + 9i = -1 + 13i$$

(iii) Multiplication

Multiplication of two complex numbers

$z_1 = a + ib$ and $z_2 = c + id$ is defined as

$$z_1z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Illustration: If $z_1 = 4 + 3i$ and $z_2 = 3 - 2i$, then

$$z_1z_2 = (4 + 3i)(3 - 2i)$$
$$= [4 \times 3 - 3 \times (-2)] + i[4 \times (-2) + 3 \times 3] = 18 + i$$

Example 5: Find the product of $2 - 3i$ and $7 + 5i$.

Solution: $(2 - 3i)(7 + 5i) = 2(7 + 5i) - 3i(7 + 5i)$

$$= 14 + 10i - 21i - 15i^2$$
$$= 14 - 11i - 15(-1) \quad (\because i^2 = -1)$$
$$= 14 - 11i + 15 = 29 - 11i$$

(iv) Division

The division of one complex number by another complex number can not be carried out, because the denominator consists of two independent terms. This difficulty can be overcome by multiplying the numerator and denominator by the conjugate of the complex number in the denominator. This process is called **rationalization.**

We have $\frac{z_1}{z_2} = \frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$ (By rationalization)

$$= \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac - adi + bci - bdi^2}{c^2 + d^2}$$

$$= \frac{(ac + bd) - (ad - bc)i}{c^2 + d^2} \quad (\because i^2 = -1)$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.$$ Thus $\frac{z_1}{z_2} = \frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$

Illustration: Solve $(x+iy)(2-3i)=4+i$, where x and y are real.

Solution: We have, $(x+iy)(2-3i)=4+i$

$$\Rightarrow x+iy=\frac{4+i}{2-3i}=\frac{4+i}{2-3i}\times\frac{2+3i}{2+3i}=\frac{(8-3)+i(12+2)}{2^2-(3i)^2}=\frac{5+14i}{13}=\frac{5}{13}+\frac{14}{13}i$$

$$\therefore \quad x=\frac{5}{13} \quad \text{and} \quad y=\frac{14}{13}$$

Example 6: Write $\frac{3+2i}{4-3i}$ in the form $a+bi$.

Solution:

$$\frac{3+2i}{4-3i}=\frac{3+2i}{4-3i}\times\frac{4+3i}{4+3i} \qquad \textit{(By rationalization)}$$

$$=\frac{(3+2i)(4+3i)}{(4-3i)(4+3i)}=\frac{12+9i+8i+6i^2}{16+12i-12i-9i^2}=\frac{12+17i+6(-1)}{16-9(-1)} \quad (\because i^2=-1)$$

$$=\frac{6+17i}{25}=\frac{6}{25}+\frac{17}{25}i$$

1.1.5 Absolute value or modulus of a complex number

Let $z=(a,b)=a+bi$ be a complex number. Then **absolute value** (or **modulus**) of z, denoted by $|z|$, is defined by $|z|=\sqrt{a^2+b^2}$.

In the adjoining figure P represents $a+bi$. $\overline{PM}$ is a perpendicular drawn on $\overline{OX}$

Therefore $\overline{OM}=a$ and $\overline{PM}=b$. In the right angled–triangle OMP, we have by Pythagoras theorem.

$$\left|\overline{OP}\right|^2=\left|\overline{OM}\right|^2+\left|\overline{PM}\right|^2=a^2+b^2$$

$$\therefore \left|\overline{OP}\right|=\sqrt{a^2+b^2}=|z|.$$ Thus, the modulus of a complex number is the distance from the origin to the point representing the number.

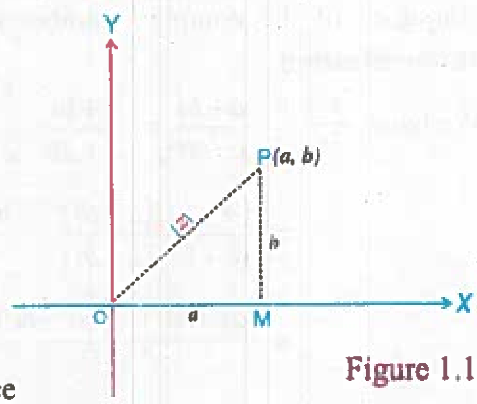


Figure 1.1

Example 7: Compute the absolute value of the given complex numbers:

(a) i (b) 3 (c) $2-5i$

Solution: (a) Let $z = i$ or $z = 0 + i$,

Then by definition

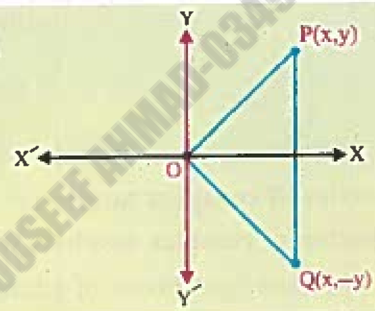
$$|z| = \sqrt{(0)^2 + (1)^2} = \sqrt{1^2} = 1$$

(b) Let $z = 3$ or $z = 3 + 0i$. Then $|z| = \sqrt{(3)^2 + (0)^2} = \sqrt{9} = 3$

(c) Let $z = 2 - 5i$. Then $|z| = \sqrt{(2)^2 + (-5)^2} = \sqrt{4+25} = \sqrt{29}$

For Your Information

The complex number $z = x + iy$ and its conjugate $\overline{z} = x - iy$ are respectively represented by the points $P(x, y)$ and $Q(x, -y)$. Geometrically, the point $Q(x, -y)$ is the mirror image of the point $P(x, y)$ on the x-axis and vice versa.



EXERCISE 1.1

1. Simplify the following

(i) $i^9 + i^{19}$ (ii) $(-i)^{23}$ (iii) $(-1)^{\frac{-23}{2}}$ (iv) $(-1)^{\frac{15}{2}}$

2. Prove that $i^{107} + i^{112} + i^{122} + i^{153} = 0$

3. Add the following complex numbers

(i) $3(1+2i), -2(1-3i)$ (ii) $\frac{1}{2}-\frac{2}{3}i,\ \frac{1}{4}-\frac{1}{3}i$ (iii) $\left(\sqrt{2}, 1\right), \left(1, \sqrt{2}\right)$

4. Subtract the second complex number from first

(i) $(a, 0), (2, -b)$ (ii) $(-3, \frac{1}{2}), (3, \frac{1}{2})$ (iii) $3\sqrt{3}-5\sqrt{7}i,\ \sqrt{3}+2\sqrt{7}i$

5. Multiply the following complex numbers

(i) $8i + 11,\ -7+5i$ (ii) $3i, 2(1-i)$ (iii) $\sqrt{2}+\sqrt{3}\,i,\ 2\sqrt{2}-\sqrt{3}\,i$

6. Perform the indicated division and write the answer in the form $a+bi$

(i) $\frac{4+i}{3+5i}$ (ii) $\frac{1}{-8+i}$ (iii) $\frac{1}{7-3i}$ (iv) $\frac{6+i}{i}$

7. If $z_1 = 1 + 2i$ and $z_2 = 2+3i$, evaluate

(i) $|z_1 + z_2|$ (ii) $|z_1 z_2|$ (iii) $\left|\frac{z_1}{z_2}\right|$

8. Express the following in the standard form $a + ib$

(i) $\frac{1-2i}{2+i}+\frac{4-i}{3+2i}$ (ii) $\frac{2+\sqrt{-9}}{-5-\sqrt{-16}}$ (iii) $\frac{(1+i)^2}{4+3i}$

9. Find the conjugate of $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ 10. Evaluate $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$

11. Let $z_1 = 2 - i$, $z_2 = -2 + i$, find

(i) Re $\left(\frac{z_1 z_2}{\bar{z}_1}\right)$ (ii) Im $\left(\frac{1}{z_1 \bar{z}_1}\right)$

1.2 Properties of complex numbers

1.2.1 Properties of complex numbers with respect to addition and multiplication

Like real numbers, properties of addition and multiplication also hold in complex numbers.

(i) Properties of Addition

A-1 Addition is commutative i.e. $z_1+z_2=z_2+z_1$

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$\begin{aligned} z_1 + z_2 &= (a+bi) + (c+di) \\ &= (a+c) + (b+d)i \\ &= (c+a) + (d+b)i \quad \text{(by commutative property for addition in } \mathbb{R}) \\ &= (c+di) + (a+bi) = z_2 + z_1 \end{aligned}$$

Thus $z_1 + z_2 = z_2 + z_1$

Example 8: If $z_1 = 1 + 3i$ and $z_2 = 3 - 5i$, then $z_1 + z_2 = z_2 + z_1$

Solution:
$$\begin{aligned} z_1 + z_2 &= (1 + 3i) + (3 - 5i) \\ &= (1 + 3) + (3 - 5)i = 4 - 2i \end{aligned}$$

and
$$\begin{aligned} z_2 + z_1 &= (3 - 5i) + (1 + 3i) \\ &= (3 + 1) + (-5 + 3)i = 4 - 2i \end{aligned}$$

Hence $z_1 + z_2 = z_2 + z_1$

A–2 Addition is associative i.e. $z_1+(z_2+z_3) = (z_1+z_2) + z_3$

If $z_1 = a + bi$, $z_2 = c + di$ *and* $z_3 = e + fi$, *then*

$$\begin{aligned} z_1 + (z_2+z_3) &= (a+bi) + [(c+di) + (e+fi)] \\ &= (a+bi) + [(c+e) + (d+f)i] \\ &= a+(c+e) + [b + (d+f)]i \\ &= (a+c) + e + [(b+d) + f]i \quad \text{(by associative property for addition in } \mathbb{R}) \\ &= [(a+c) + (b+d)i] + e + fi \\ &= [(a+bi) + (c+di)] + e + fi \\ &= (z_1+z_2) + z_3 \end{aligned}$$

Thus $z_1 + (z_2+z_3) = (z_1+z_2) + z_3$

Example 9: If $z_1 = 1+2i$, $z_2 = -2+3i$ and $z_3 = -3-5i$,

then $z_1 + (z_2+z_3) = (z_1+z_2) + z_3$

Solution:

$$\begin{aligned} z_1 + (z_2+z_3) &= (1+2i) + [(-2+3i) + (-3-5i)] \\ &= (1+2i) + [(-2-3) + (3-5)i] = (1+2i) + (-5-2i) \\ &= (1-5) + (2-2)i = -4 + 0i \\ (z_1+z_2) + z_3 &= [(1+2i) + (-2+3i)] + (-3-5i) \\ &= [(1-2) + (2+3)i] + (-3-5i) = (-1+5i) + (-3-5i) \\ &= (-1-3) + (5-5)i = -4 + 0i \end{aligned}$$

Hence $z_1 + (z_2+z_3) = (z_1+z_2) + z_3$

(ii) Properties of Multiplication

M-1 Multiplication is commutative i.e. $z_1 z_2 = z_2 z_1$

If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 z_2 = (a+bi).(c+di)$

$= (ac - bd) + (ad + bc)i$ (by definition of multiplication of complex numbers)

and $z_2 z_1 = (c+di)(a+bi) = (ca - db) + (cb + da)i$

$= (ac - bd) + (ad + bc)i$ (by commutative properties of multiplication and addition of real numbers). Thus, $z_1 z_2 = z_2 z_1$

Example 10: If $z_1 = 2-3i$ and $z_2 = -1+2i$, then $z_1 z_2 = z_2 z_1$

Solution:

$$\begin{aligned} z_1 z_2 &= (2-3i)(-1+2i) = 2(-1+2i) - 3i(-1+2i) \\ &= -2 + 4i + 3i - 6i^2 = -2 + 7i + 6 \quad (\because i^2 = -1) \\ &= 4 + 7i \end{aligned}$$

and

$$\begin{aligned} z_2 z_1 &= (-1+2i)(2-3i) = -1(2-3i) + 2i(2-3i) \\ &= -2 + 3i + 4i - 6i^2 = -2+7i+6 \quad (\because i^2 = -1) \\ &= 4+7i \end{aligned}$$

Hence $z_1 z_2 = z_2 z_1$

M-2 Multiplication is associative i.e. $z_1(z_2 z_3)=(z_1 z_2)z_3$

If $z_1 = a + bi$, $z_2 = c + di$ and $z_3 = e + fi$, then

$$z_1(z_2 z_3) = (a + bi)\{(c + di)(e + fi)\} = (a + bi)\{(ce - df) + (cf + de)i\}$$
$$= [a(ce - df) - b(cf + de)] + [a(cf + de) + b(ce - df)]\,i$$

and $$(z_1 z_2)z_3 = [(a + bi)(c + di)](e + fi) = [(ac - bd) + (ad + bc)i](e + fi)$$
$$= [(ac - bd)e - (ad + bc)f] + [(ac - bd)f + (ad + bc)e]i$$
$$= [(ace - adf - (bde + bcf)] + [(acf + ade) + (bce - bdf)]i$$
$$= [a(ce - df) - b(cf + de)] + [a(cf + de) + b(ce - df)]i$$

Thus, $z_1(z_2 z_3) = (z_1 z_2) z_3$

Example 11: If $z_1 = 1 - i$, $z_2 = -1 + 2i$ and $z_3 = 2 - 3i$, then $z_1(z_2 z_3) = (z_1 z_2)z_3$

Solution: We have

$$z_1(z_2 z_3) = (1 - i)[(-1 + 2i)(2 - 3i)] = (1 - i)[-1(2 - 3i) + 2i(2 - 3i)]$$
$$= (1 - i)(-2 + 3i + 4i - 6i^2) = (1 - i)(-2 + 7i + 6)$$
$$= (1 - i)(4 + 7i) = 1(4 + 7i) - i(4 + 7i)$$
$$= 4 + 7i - 4i - 7i^2 = 4 + 3i + 7 = 11 + 3i$$

and $$(z_1 z_2)z_3 = [(1 - i)(-1 + 2i)](2 - 3i) = [1(-1 + 2i) - i(-1 + 2i)](2 - 3i)$$
$$= (-1 + 2i + i - 2i^2)(2 - 3i) = (-1 + 3i + 2)(2 - 3i)$$
$$= (1 + 3i)(2 - 3i) = 1(2 - 3i) + 3i(2 - 3i)$$
$$= 2 - 3i + 6i - 9i^2 = 2 + 3i + 9 = 11 + 3i$$

Hence $z_1(z_2 z_3) = (z_1 z_2)z_3$

(iii) Multiplication–Addition Property (The Distributive Property)

This property is more explicitly stated as follows:

M–A. Multiplication is distributive over addition i.e. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

If $z_1 = a + bi$, and $z_2 = c + di$ and $z_3 = e + fi$, then

$$z_1(z_2 + z_3) = (a + bi)[(c + di) + (e + fi)]$$
$$= (a + bi)[(c + e) + (d + f)i]$$
$$= [a(c + e) - b(d + f)] + [a(d + f) + b(c + e)]\,i$$

and $$z_1 z_2 + z_1 z_3 = (a + bi)(c + di) + (a + bi)(e + fi)$$
$$= [(ac - bd) + (ad + bc)i] + [(ae - bf) + (af + be)i]$$
$$= [(ac + ae) + (-bd - bf)] + [(ad + af)i + (bc + be)i]$$
$$= [a(c + e) - b(d + f)] + [a(d + f) + b(c + e)]i$$

Thus, $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Example 12: If $z_1 = -1 + 2i$, $z_2 = 3 + 4i$ and $z_3 = -2 + 5i$, then

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

Solution: We have

$$\begin{aligned} z_1(z_2+z_3) &= (-1+2i)\,[(3+4i)+(-2+5i)] \\ &= (-1+2i)\,[3+4i-2+5i] \\ &= (-1+2i)(1+9i) = -1-9i+2i+18i^2 \\ &= -1-7i-18 = -19-7i \quad (\because i^2=-1) \end{aligned}$$

and $z_1z_2 + z_1z_3 = (-1+2i)(3+4i) + (-1+2i)(-2+5i)$

$$= -3-4i+6i+8i^2+2-5i-4i+10i^2$$

$$= -1-7i-8-10 = -19-7i$$

Hence $z_1(z_2+z_3) = z_1z_2 + z_1z_3$

Note

According to the commutative property for multiplication $iy = yi$. Hence we can write $z = x + iy$ instead of $z = x + yi$

1.2.2 Additive identity and multiplicative identity of complex numbers

A complex number $c + di$ is called the **additive identity** of the complex number $a + bi$ if $(a+bi)+(c+di) = (c+di)+(a+bi) = a+bi$

Let $a + bi$ be any complex number and $c + di = 0 + 0i$ be the zero complex number. Then

$$(a+bi)+(0+0i) = (a+0)+(b+0)i \quad \text{(by definition of addition)}$$

$$= a + bi$$

Similarly $(0+0i)+(a+bi) = a+bi$

Thus the additive identity in **C** is the zero complex number i.e. $0 + 0i$

A complex number $c + di$ is called the **multiplicative identity** of the complex number $a + bi$ if $(a+bi)(c+di) = (c+di)(a+bi) = a+bi$

Let $a + bi$ be any complex number and $c + di = 1 + 0i$ be the unit complex number. Then

$(a+bi)(1+0i) = (a\cdot 1 - b\cdot 0) + (a\cdot 0 + b\cdot 1)i$ (by definition of multiplication of complex numbers)

$$= a + bi$$

Similarly $(1+0i)(a+bi) = a+bi$

Thus the multiplicative identity in **C** is the unit complex number $1 + 0i$.

1.2.3 Additive inverse and multiplicative inverse of complex numbers

A complex number $c + di$ is called the **additive inverse** of the complex number $a+bi$ if $(a+bi)+(c+di) = 0+0i$ i.e. the additive identity.

We have $(a+bi)+(c+di) = 0+0i \Rightarrow (a+c)+(b+d)i = 0+0i$

$\Rightarrow \quad a+c = 0$ and $b+d = 0 \quad \Rightarrow \quad c = -a$ and $d = -b$

so that $c + di = -a - bi$, Thus the additive inverse of $a + bi$ is $-a - bi$.

Example 13: Find additive inverse of $5-3i$

Solution:

Let $z=5-3i$

$\therefore -z=-(5-3i)=-5+3i$

Thus the additive inverse of $5-3i$ is $-5+3i$.

Multiplicative inverse A complex number $c+di$ is called the **multiplicative inverse** of the complex number $a+bi$ if $(a+bi)(c+di)=1+0i$ i.e. the multiplicative identity.

We have $(a+bi)(c+di)=1+0i \Rightarrow (ac-bd)+(ad+bc)i=1+0i$

$\Rightarrow \quad ac-bd=1 \quad$ (i)

and $\quad ad+bc=0 \quad$ (ii) From (ii), we have

$ad=-bc$ or $d=-\dfrac{bc}{a}$ (iii) Putting the value of d in (i), we get

$$ac+b\left(\frac{bc}{a}\right)=1 \Rightarrow \frac{a^2c+b^2c}{a}=1$$

$$\Rightarrow (a^2+b^2)c=a \Rightarrow c=\frac{a}{a^2+b^2} \quad \text{(iv)}$$

Putting the value of c in (iii), we get

$$d=-\frac{b\cdot a}{a(a^2+b^2)} \Rightarrow d=-\frac{b}{a^2+b^2} \quad \text{(v)}$$

From (iv) and (v), we have

$$c+di=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$$

Thus the multiplicative inverse of $a+bi$ is

$$\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$$

Did You Know

The complex numbers possess all the properties that real numbers possess except for the order relation, that is, we can not say that one complex number is greater than the other complex number.

Example 14: Find multiplicative inverse of $-2-3i$

Solution: Let $z=-2-3i$ Here $a=2$, $b=-3$

$$\therefore z^{-1}=\left(\frac{a}{a^2+b^2},-\frac{b}{a^2+b^2}\right)=\left(\frac{-2}{(-2)^2+(-3)^2},-\frac{-3}{(-2)^2+(-3)^2}\right)=\left(\frac{-2}{13},\frac{3}{13}\right)=\frac{-2}{13}+\frac{3}{13}i$$

Thus $-\dfrac{2}{13}+\dfrac{3}{13}i$ is the multiplicative inverse of $-2-3i$.

1.2.4 Some properties of the conjugate and modulus of complex numbers

In the following theorem we prove some properties pertaining to conjugation and modulus of complex numbers.

Theorem: For all z_1, z_2, z_3 in **C**

(a) $|z| = |-z| = |\bar{z}| = |-\bar{z}|$ (b) $\bar{\bar{z}} = z$ (c) $z\bar{z} = |z|^2$

(d) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (e) $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$ (f) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

Proof (a) Let $z = a + bi$. Then $-z = -a - bi$, $\bar{z} = a - bi$ and $-\bar{z} = -a + bi$

Therefore by definition

$$|z| = \sqrt{a^2 + b^2} \quad \text{(i)}$$

$$|-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} \quad \text{(ii)}$$

$$|\bar{z}| = \sqrt{(a)^2 + (-b)^2} = \sqrt{a^2 + b^2} \quad \text{(iii)}$$

$$|-\bar{z}| = \sqrt{(-a)^2 + (b)^2} = \sqrt{a^2 + b^2} \quad \text{(iv)}$$

Equation (i), (ii), (iii) and (iv) yield that

$$|z| = |-z| = |\bar{z}| = |-\bar{z}|$$

(b) Let $z = a + bi$, then $\bar{z} = a - bi$, and so

$$\bar{\bar{z}} = a + bi = z$$

Thus $\bar{\bar{z}} = z$

(c) Let $z = a + bi$. Then $\bar{z} = a - bi$

Therefore

$$z\bar{z} = (a + bi)(a - bi)$$
$$= a^2 - abi + bai - b^2i^2$$
$$= a^2 - (-1)b^2 \quad (\because i^2 = -1)$$
$$= a^2 + b^2$$
$$= |z|^2 \quad (\because |z| = \sqrt{a^2 + b^2})$$

Thus $z\bar{z} = |z|^2$

(d) Let $z_1 = a + bi$ and $z_2 = c + di$

Then $\bar{z}_1 = a - bi$, $\bar{z}_2 = c - di$ and

$$z_1 + z_2 = (a + bi) + (c + di)$$
$$= (a + c) + (b + d)i$$

Therefore $\overline{z_1 + z_2} = (a + c) - (b + d)i$

$$= (a - bi) + (c - di) = \bar{z}_1 + \bar{z}_2$$

Thus $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(e) Let $z_1 = a + bi$ and $z_2 = c + di$

Then $\overline{z_1 z_2} = \overline{(a+bi)(c+di)}$

$$= \overline{(ac-bd)+(ad+bc)i}$$

$$= (ac - bd) - (ad + bc)\, i \quad \text{(i)}$$

and $\bar{z}_1 \bar{z}_2 = \overline{(a+bi)}\ \overline{(c+di)} = (a - bi)(c - di)$

$$= (ac - bd) + (-ad - bc)\, i$$

$$= (ac - bd) - (ad + bc)\, i \quad \text{(ii)}$$

Thus from equations (i) and (ii),we have

$$\overline{z_1 z_2} = \bar{z}_1\, \bar{z}_2$$

(f) Let $z_1 = a + bi$ and $z_2 = c + di$

Then $\dfrac{z_1}{z_2} = \dfrac{a+bi}{c+di} = \dfrac{a+bi}{c+di} \times \dfrac{c-di}{c-di}$ (by rationalization)

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$$

$$\therefore \quad \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i}$$

$$= \frac{ac+bd}{c^2+d^2} - \frac{bc-ad}{c^2+d^2} i \quad \text{(i)}$$

and $\dfrac{\bar{z}_1}{\bar{z}_2} = \dfrac{\overline{a+bi}}{\overline{c+di}} = \dfrac{a-bi}{c-di} = \dfrac{a-bi}{c-di} \times \dfrac{c+di}{c+di}$ (by rationalization)

$$= \frac{(ac+bd)-(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} - \frac{bc-ad}{c^2+d^2} i \quad \text{(ii)}$$

Thus from equations (i) and (ii), we have

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

1.2.5. Real and imaginary parts of the complex number of the form

(i) $(x+iy)^n$ (ii) $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n$, $x_2+iy_2 \neq 0$ where $n = \pm 1$ and ± 2

i. Real and imaginary parts of $(x+iy)^n$ where $n = \pm 1$ and ± 2

when $n = 1$, $(x+iy)^n$ reduces to $x+iy$

Therefore, real part $= x$ and imaginary part $= y$

When $n = -1$, $(x+iy)^n$ reduces to $(x+iy)^{-1}$

We have, $(x+iy)^{-1} = \dfrac{1}{(x+iy)} = \dfrac{1}{(x+iy)} \times \dfrac{x-iy}{x-iy}$ (by rationalization)

$$= \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Therefore real part $= \dfrac{x}{x^2+y^2}$ and imaginary part $= \dfrac{-y}{x^2+y^2}$

When $n = 2$, $(x+iy)^n$ reduces to $(x+iy)^2$,

we have $(x+iy)^2 = x^2 + 2ixy + i^2y^2$

$= x^2 + 2ixy - y^2$ $(\because i^2 = -1)$

$= (x^2 - y^2) + 2ixy$

Therefore real part $= x^2 - y^2$ and imaginary part $= 2xy$

When $n = -2$, $(x+iy)^n$ reduces to $(x+iy)^{-2}$

We have, $(x+iy)^{-2} = \dfrac{1}{(x+iy)^2}$

$$= \frac{1}{(x+iy)^2} \times \frac{(x-iy)^2}{(x-iy)^2} = \frac{x^2-y^2-2ixy}{(x+iy)^2(x-iy)^2}$$

$$= \frac{x^2-y^2-2ixy}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} - i\frac{2xy}{(x^2+y^2)^2}$$

Therefore real part $= \dfrac{x^2-y^2}{(x^2+y^2)^2}$ and imaginary part $= \dfrac{-2xy}{(x^2+y^2)^2}$

Example 15: Find the real and imaginary parts of the following complex numbers.

(i) $2-3i$ (ii) $(5-3i)^{-1}$ (iii) $(3+i)^2$ (iv) $(1+2i)^{-2}$

Solution:

(i) Let $z = 2-3i$ Therefore real part of $z = 2$ and imaginary part of $z = -3$

(ii) Let $z = (5-3i)^{-1}$. Here $x = 5$ and $y = -3$

$$\text{Therefore, real part of } z = \frac{x}{x^2+y^2} = \frac{5}{(5)^2+(-3)^2} = \frac{5}{25+9} = \frac{5}{34}$$

$$\text{and imaginary part of } z = \frac{-y}{x^2+y^2} = \frac{-(-3)}{(5)^2+(-3)^2} = \frac{3}{25+9} = \frac{3}{34}$$

(iii) Let $z = (3+i)^2$. Here $x = 3$ and $y = 1$

Therefore, real part of $z = x^2-y^2 = (3)^2-(1)^2 = 9-1 = 8$

imaginary part of $z = 2xy = 2(3)(1) = 6$

(iv) Let $z = (1+2i)^{-2}$. Here $x = 1$ and $y = 2$

$$\text{Therefore, real part of } z = \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{(1)^2-(2)^2}{[(1)^2+(2)^2]^2} = \frac{1-4}{(5)^2} = \frac{-3}{25}$$

$$\text{imaginary part of } z = \frac{-2xy}{(x^2+y^2)^2} = \frac{-2(1)(2)}{[(1)^2+(2)^2]} = \frac{-4}{(5)^2} = \frac{-4}{25}$$

ii. Real and imaginary parts of $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n$ where $n = \pm 1$ and ± 2

When $n = 1$, $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n$ reduces to $\frac{x_1+iy_1}{x_2+iy_2}$. We have,

$$\frac{x_1+iy_1}{x_2+iy_2} = \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_2-iy_2}{x_2-iy_2} \quad \text{(By rationalization)}$$

$$= \frac{x_1x_2-ix_1y_2+iy_1x_2-i_2y_1y_2}{x_2^2-i^2y_2^2} = \frac{x_1x_2+i(y_1x_2-x_1y_2)+y_1y_2}{x_2^2+y_2^2} \quad (\because i^2 = -1)$$

$$= \frac{(x_1x_2+y_1y_2)+i(y_1x_2-x_1y_2)}{x_2^2+y_2^2} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + i\frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}$$

$$\text{Therefore, real part} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} \text{ and imaginary part} = \frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}$$

When $n = -1$, $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n$ reduces to $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^{-1}$

We have $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^{-1} = \dfrac{x_2+iy_2}{x_1+iy_1} = \dfrac{x_2+iy_2}{x_1+iy_1} \times \dfrac{x_1-iy_1}{x_1-iy_1}$ (by rationalization)

$$= \frac{x_2\,x_1 + y_2\,y_1}{x_1^2+y_1^2} + i\,\frac{y_2\,x_1 - x_2\,y_1}{x_1^2+y_1^2} \text{ (by routine calculation)}$$

Therefore, real part $= \dfrac{x_2\,x_1 + y_2\,y_1}{x_1^2+y_1^2}$ and imaginary part $= \dfrac{y_2\,x_1 - x_2\,y_1}{x_1^2+y_1^2}$

When $n = 2$, $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n$ reduces to $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^2$. We have,

$$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^2 = \frac{(x_1+iy_1)^2}{(x_2+iy_2)^2} = \frac{(x_1+iy_1)^2}{(x_2+iy_2)^2} \times \frac{(x_2-iy_2)^2}{(x_2-iy_2)^2} \text{ (By rationalization)}$$

$$= \frac{[(x_1^2-y_1^2)+2ix_1y_1]\;[(x_2^2-y_2^2)-2ix_2y_2]}{(x_2+iy_2)^2\,(x_2-iy_2)^2}$$

$$= \frac{[(x_1^2-y_1^2)\,(x_2^2-y_2^2)+4x_1x_2y_1y_2\;]+2i\,[x_1y_1(x_2^2-y_2^2)-x_2y_2\,(x_1^2-y_1^2)]}{(x_2^2+y_2^2)^2}$$

Therefore, real part $= \dfrac{(x_1^2-y_1^2)\,(x_2^2-y_2^2)+4x_1x_2y_1y_2}{(x_2^2+y_2^2)^2}$

imaginary part $= 2\,\dfrac{x_1y_1(x_2^2-y_2^2)-x_2y_2\,(x_1^2-y_1^2)}{(x_2^2+y_2^2)^2}$

When $n = -2$, $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^n$ reduces to $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^{-2}$

We have $\left(\dfrac{x_1+iy_1}{x_2+iy_2}\right)^{-2} = \dfrac{(x_2+iy_2)^2}{(x_1+iy_1)^2} = \dfrac{(x_2+iy_2)^2}{(x_1+iy_1)^2} \times \dfrac{(x_1-iy_1)^2}{(x_1-iy_1)^2}$

$$=\frac{[(x_2^2-y_2^2)(x_1^2-y_1^2)+4x_1x_2y_1y_2]+2i[x_2y_2(x_1^2-y_1^2)-x_1y_1(x_2^2-y_2^2)]}{(x_1^2+y_1^2)^2}$$

Therefore, real part $=\dfrac{(x_2^2-y_2^2)(x_1^2-y_1^2)+4x_1x_2y_1y_2}{(x_1^2+y_1^2)^2}$

imaginary part $=2\dfrac{x_2y_2(x_1^2-y_1^2)-x_1y_1(x_2^2-y_2^2)}{(x_1^2+y_1^2)^2}$

EXERCISE 1.2

1. If $z_1 = 2 + i$ and $z_2 = 1-i$, then verify commutative property w.r.t. addition and multiplication.
2. $z_1 = -1+i$, $z_2 = 3-2i$ and $z_3 = 2+3i$, verify associative property w.r.t. addition and multiplication.
3. $z_1=\sqrt{3}+\sqrt{2}i$, $z_2=\sqrt{2}-\sqrt{3}i$ and $z_3=2-2i$, verify distributive property of multiplication over addition.
4. Find the additive and multiplicative inverses of the following complex numbers.
   (i) $5+2i$ (ii) $(7,-9)$
5. (i) Let $z_1=2+4i$ and $z_2=1-3i$. Verify that $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$
   (ii) Let $z_1=2+3i$ and $z_2=2-3i$. Verify that $\overline{z_1z_2}=\overline{z_1}\,\overline{z_2}$
   (iii) If $z_1=-a-3bi$, $z_2=2a-3bi$, then verify that $\overline{\left(\dfrac{z_1}{z_2}\right)}=\dfrac{\overline{z_1}}{\overline{z_2}}$
6. Show that for all complex numbers z_1 and z_2
   (i) $|z_1z_2|=|z_1||z_2|$ (ii) $\left|\dfrac{z_1}{z_2}\right|=\dfrac{|z_1|}{|z_2|}$, where $z_2\neq 0$.
7. Separate into real and imaginary parts
   (i) $\dfrac{2+3i}{5-2i}$ (ii) $\dfrac{(1+2i)^2}{1-3i}$ (iii) $\dfrac{1-i}{(1+i)^2}$
   (iv) $(2a-bi)^{-2}$ (v) $\left(\dfrac{3+4i}{4-3i}\right)^{-2}$ (vi) $\left(\dfrac{4-5i}{2+3i}\right)^{2}$

8. Show that

(i) $z+\overline{z}=2Re(z)$ (ii) $z-\overline{z}=2i\,Im(z)$

(iii) $z\overline{z}=[Re(z)]^2+[Im(z)]^2$ (iv) $z=\overline{z}\Rightarrow z$ is real

(v) $\overline{z}=-z$ if and only if z is pure imaginary

9. If $z=3+2i$, then verify that (i) $-|z|\le Re(z)\le|z|$ (ii) $-|z|\le Im(z)\le|z|$

1.3 Solution of equations

In this section we shall find solution of different equations in complex variables either with real or complex coefficients.

1.3.1 Solution of simultaneous linear equations with complex coefficients

Consider the following equation

$$pz+qw=r \quad (1)$$

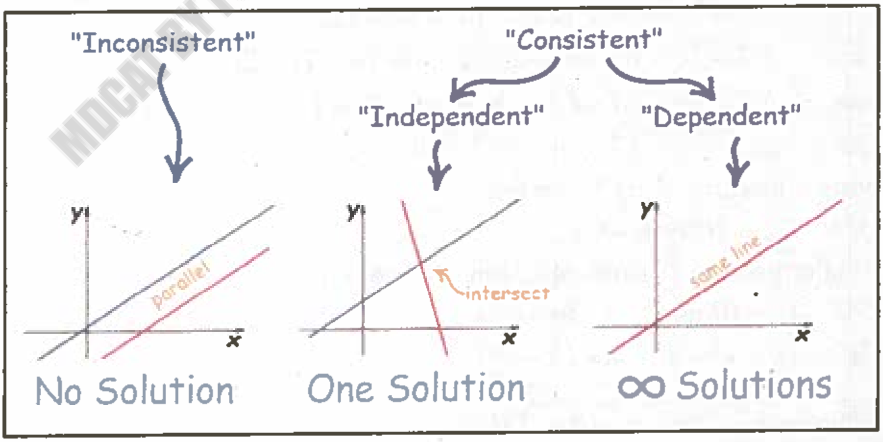
where p,q and r are complex numbers. The equation (1) is called a **linear equation** in two complex variables (or unknown) z and w.

$$\left.\begin{aligned} p_1 z + q_1 w &= r_1 \\ p_2 z + q_2 w &= r_2 \end{aligned}\right\} \quad (2)$$

These two equations together form a system of linear equations in two variables z and w. The linear equations in two variables are also called **simultaneous linear equations**.

For example
$$\left.\begin{aligned} 5z-(3+i)w &= 7-i \\ (2-i)z+2iw &= -1+i \end{aligned}\right\} \quad (3)$$

is a system of linear equations with complex coefficients.



A **solution** of a system in two variables z and w is an ordered pair (z,w) such that both the equations in the system are satisfied. For example consider system (3). The ordered pair (z,w) where $z = 1+i$ and $w = 2i$ is a solution of (3) because if we replace z by $1+i$ and w by $2i$, then both the equations are satisfied. The process of finding all solutions of the system of equations is called **solving** the system.

Here we shall find solution of a system of two equations with complex co-efficient in two variables z and w. The simple rule for solving such system of equations is the "method of elimination and substitution".

Step–1 If necessary multiply each equation by a constant so that the co-efficient of one variable in each equation is the same.

Step–2 Add or subtract the resulting equations to eliminate one variable, thus getting an equation in one variable.

Step–3 Solve the equation in one variable obtained in Step-2.

Step–4 Substitute the known value of one variable in either of the original equations in step-1 and solve for the other variable.

Step–5 Writing together the corresponding values of the variables in the form of ordered pairs gives solution of the system.

Example 16: Solve the simultaneous linear equations with complex coefficients.

$$5z - (3+i)w = 7 - i$$
$$(2-i)z + 2iw = -1 + i$$

Solution: Given that

$$5z - (3+i)w = 7 - i \qquad (1)$$
$$(2-i)z + 2iw = -1 + i \qquad (2)$$

Multiplying equation (1) by $(2-i)$ we have

$$5(2-i)z - (3+i)(2-i)w = (7-i)(2-i)$$
$$\Rightarrow \quad 5(2-i)z - (6 - 3i + 2i - i^2)w = 14 - 7i - 2i + i^2$$
$$\Rightarrow \quad 5(2-i)z - (6 - i + 1)w = 14 - 9i - 1 \quad (\because i^2 = -1)$$
$$\Rightarrow \quad 5(2-i)z - (7-i)w = 13 - 9i \qquad (3)$$

Multiplying equation (2) by 5, we have

$$5(2-i)z + 10iw = -5 + 5i \qquad (4)$$

Subtracting equation (3) from equation (4), we have

$$\begin{array}{l} 5(2-i)z + 10iw \qquad = -5 + 5i \\ +5(2-i)z - (7-i)w = +13 - 9i \\ \;\;- \qquad\quad + \qquad\qquad - \quad\; + \\ \hline 10iw + (7-i)w \;= -18 + 14i \end{array}$$

$$\Rightarrow \quad (7+9i)w = -18 + 14i \quad \Rightarrow w = \frac{-18+14i}{7+9i}$$

$$\Rightarrow \quad w = \frac{-18+14i}{7+9i} \times \frac{7-9i}{7-9i} \quad \text{(By Rationalization)}$$

$$\Rightarrow \quad w = \frac{260i}{130} = 2i$$

Substituting the value of w in (1), we have

$$5z-(3+i)(2i) = 7-i \Rightarrow \quad 5z-(6i+2i^2) = 7-i$$

$$\Rightarrow \quad 5z-(6i-2) = 7-i \quad \Rightarrow \quad 5z = 7-i+6i-2$$

$$\Rightarrow \quad 5z = 5+5i \quad \Rightarrow \quad z = \frac{5+5i}{5} = 1+i$$

Thus (z,w) where $z = 1+i$ and $w = 2i$ is the solution of the simultaneous linear equations.

1.3.2 Expression of the polynomial P(z) as a product of linear factors

Recall that an expression of the form

$P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots\ldots + a_1x + a_0,\ a_n \neq 0$

where n is a positive integer or zero and the coefficients $a_n, a_{n-1}, \ldots, a_1$ and a_0 are constants that are either to be real or complex numbers, is a polynomial of **degree n**.

For example, $2x+3$, $3x^2+2x+1$ and $5x^3-6x^2+5x-1$ are polynomials of degree 1, 2 and 3 respectively.

Here we are concerned with finding the linear factors of the following two types of polynomials.

(i) $P(z)=z^2+a^2$, where a is a real number.

(ii) $P(z)=az^3+bz^2+cz+d$ where a, b, c and d are real numbers.

In factorizing polynomials of type (i) we simply use the fact that $i^2=-1$ so that to find linear factors.

For example, $P(z) = z^2+a^2 = z^2-i^2a^2 = (z+ia)(z-ia)$. However, in factorizing polynomials of type (ii), we use the factor theorem which has already been proved in the previous class and stated below.

The factor theorem: Let $P(x)$ be any polynomial. Then $x-a$ is a factor of $P(x)$ if and only if $P(a)=0$

The method for factorizing the polynomials of type (ii) into linear factors is explained through the following example.

Example 17: Factorize the polynomial $P(z) = z^3 + 5z^2 + 19z - 25$ into linear factors.

Solution: In factorizing the given polynomial $P(z)$ into linear factors, we use the factor theorem. To do so, we note that $z = 1$ is a root of $P(z)$, since

$$P(1) = (1)^3 + 5(1)^2 + 19(1) - 25 = 1 + 5 + 19 - 25 = 0$$

By factor theorem $z - 1$ is a factor of $P(z)$. We therefore arrange the terms in such a way that we can find a common factor $z - 1$ as follows:

$$\begin{aligned}
P(z) &= z^3 + 5z^2 + 19z - 25 \\
&= (z^3 - 1) + (5z^2 + 19z - 24) \\
&= (z-1)(z^2 + z + 1) + (5z^2 - 5z + 24z - 24) \quad \because a^3 - b^3 = (a-b)(a^2 + ab + b^2) \\
&= (z-1)(z^2 + z + 1) + (5z^2 - 5z) + (24z - 24) \\
&= (z-1)(z^2 + z + 1) + 5z(z-1) + 24(z-1) \\
&= (z-1)[(z^2 + z + 1) + 5z + 24)] = (z-1)(z^2 + 6z + 25) \\
&= (z-1)(z^2 + 6z + 9 + 16) = (z-1)[(z^2 + 6z + 9) + 16] \\
&= (z-1)[(z^2 + 6z + 9) - (-16)] \\
&= (z-1)[(z+3)^2 - (4i)^2] \qquad (\because i^2 = -1) \\
&= (z-1)[(z+3) + 4i][(z+3) - 4i] \qquad (\because a^2 - b^2 = (a+b)(a-b)) \\
&= (z-1)(z + 3 + 4i)(z + 3 - 4i)
\end{aligned}$$

1.3.3 Quadratic equation of the form $pz^2 + qz + r = 0$

Consider the quadratic equation of the form

$$pz^2 + qz + r = 0 \qquad (1)$$

where p, q, r are real numbers $p \neq 0$ and z is a complex variable.

We see that $z^2 - z + 3 = 0$, $3z^2 - 4z + 2 = 0$, $5z^2 + 6z = 0$, $z^2 - 3 = 0$, $2z^2 = 3z - 1$ and $z^2 = 0$ are all examples of quadratic equation in the variable z. Equation (1) is called the standard form of the quadratic equation.

Solution of quadratic equations

Recall that all those values of z for which the given equation is true are called **solutions** or **roots** of the equation, and the set of all solutions is called **solution set.**

For example, $z^2 + 4 = 0$ or $z^2 - (2i)^2 = 0$ is true only for $z = 2i$ or $z = -2i$, hence $z = 2i$ and $z = -2i$ are the solutions or roots of the given quadratic equation and $\{2i, -2i\}$ is the **solution set.**

To find the solutions of equations of the form (1), we use a method known as " completing the square" which is described as follows:

Step–1 Write the quadratic equation in its standard form.

Step–2 Divide both sides of the equation by the coefficient of z^2 if it is other than 1.

Step–3 Shift the constant term to the right hand side of the equation.

Step–4 Add a number which is the square of half of the coefficient of z to both sides of the equation.

Step–5 Write the left hand side of the equation as a perfect square and simplify the right hand side.

Step–6 Take square root of both sides of the equation and solve the resulting equation to find the solutions of the equation.

The method is explained in the following example.

Example 18: Solve the quadratic equation $z^2 + 6z + 25 = 0$

Solution: We have

$z^2 + 6z + 25 = 0$ (Step–1)

$\Rightarrow$ $z^2 + 6z = -25$ (Step–2 and Step–3)

$\Rightarrow$ $z^2 + 6z + 9 = -25 + 9$ (Step–4)

$\Rightarrow$ $(z + 3)^2 = -16$ (Step–5)

$\Rightarrow$ $(z + 3)^2 = (4i)^2$

$\Rightarrow$ $z + 3 = \pm 2i$ (Step–6)

$\Rightarrow$ $z = -3 + 2i$ or $z = -3 - 2i$

Did You Know ?

The coefficient of z^2 must not be zero otherwise it becomes linear

Thus the solutions of given equation are $-3 + 2i$, $-3 - 2i$ and solution set is $\{-3 + 2i, -3 - 2i\}$

Example 19: Solve the equation $z^2 + z + 1 = 0$

Solution: According to the quadratic formula, the answer is

$$z = \frac{-1 \pm \sqrt{1^2 - 4}}{2} = -1 \pm i\frac{\sqrt{3}}{2}$$

EXERCISE 1.3

1. Solve the simultaneous linear equations with complex coefficients.

   (i) $z - 4w = 3i$
   $2z + 3w = 11 - 5i$

   (ii) $z + w = 3i$
   $2z + 3w = 2$

   (iii) $3z + (2+i)w = 11 - i$
   $(2 - i)z - w = -1 + i$

2. Factorize the polynomials *P(z)* into linear factors.

   (i) $P(z) = z^3 + 6z + 20$ (ii) $P(z) = 3z^2 + 7$

   (iii) $P(z) = z^2 + 4$ (iv) $P(z) = z^3 - 2z^2 + z - 2$

3. Show that each $z_1 = -1 + i$ and $z_2 = -1 - i$ satisfies the equation $z^2 + 2z + 2 = 0$

4. Determine whether $1 + 2i$ is a solution of $z^2 - 2z + 5 = 0$

5. Find all solutions to the following equations

   (i) $z^2 + z + 3 = 0$ (ii) $z^2 - 1 = z$ (iii) $z^2 - 2z + i = 0$ (iv) $z^2 + 4 = 0$

6. Find the solutions to the following equations

   (i) $z^4 + z^2 + 1 = 0$ (ii) $z^3 = -8$ (iii) $(z - 1)^3 = -1$ (iv) $z^3 = 1$

REVIEW EXERCISE 1

1. Choose the correct option.

   (i) $\left(\frac{2i}{1+i}\right)^2$

   (a) i (b) $2i$ (c) $1 - i$ (d) $1 - 2i$

   (ii) Divide $\frac{5+2i}{4-3i}$

   (a) $-\frac{7}{25} + \frac{26}{25}i$ (b) $\frac{5}{4} - \frac{2}{3}i$ (c) $\frac{14}{25} + \frac{23}{25}i$ (d) $\frac{26}{7} + \frac{23}{7}i$

   (iii) $i^{57} + \frac{1}{i^{25}}$ when simplified has the value

   (a) 0 (b) $2i$ (c) $-2i$ (d) 2

   (iv) $1 + i^2 + i^4 + i^6 + \ldots\ldots + i^{2n}$ is

   (a) Positive (b) negative (c) 0 (d) cannot be determined

   (v) If $z = x + iy$ and $\left|\frac{z - 5i}{z + 5i}\right| = 1$ then z lies on

   (a) X-axis (b) Y-axis (c) line $y = 5$ (d) None of these

   (vi) The multiplicative inverse of $z = 3 - 2i$, is

   (a) $\frac{1}{3}(3 + 2i)$ (b) $\frac{1}{13}(3 + 2i)$ (c) $\frac{1}{13}(3 - 2i)$ (d) $\frac{1}{4}(3 - 2i)$

(vii) If $(x + iy)(2 - 3i) = 4 + i$, then

(a) $x = -14/13, y = 5/13$ (b) $x = 5/13, y = 14/13$

(c) $x = 14/13, y = 5/13$ (d) $x = 5/13, y = -14/13$

2. Show that $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \forall\, n \in N$

3. Express the following complex numbers in the form $x + iy$.

(i) $(1+3i) + (5+7i)$ (ii) $(1+3i) - (5+7i)$ (iii) $(1+3i)(5+7i)$ (iv) $\frac{1+3i}{5+7i}$

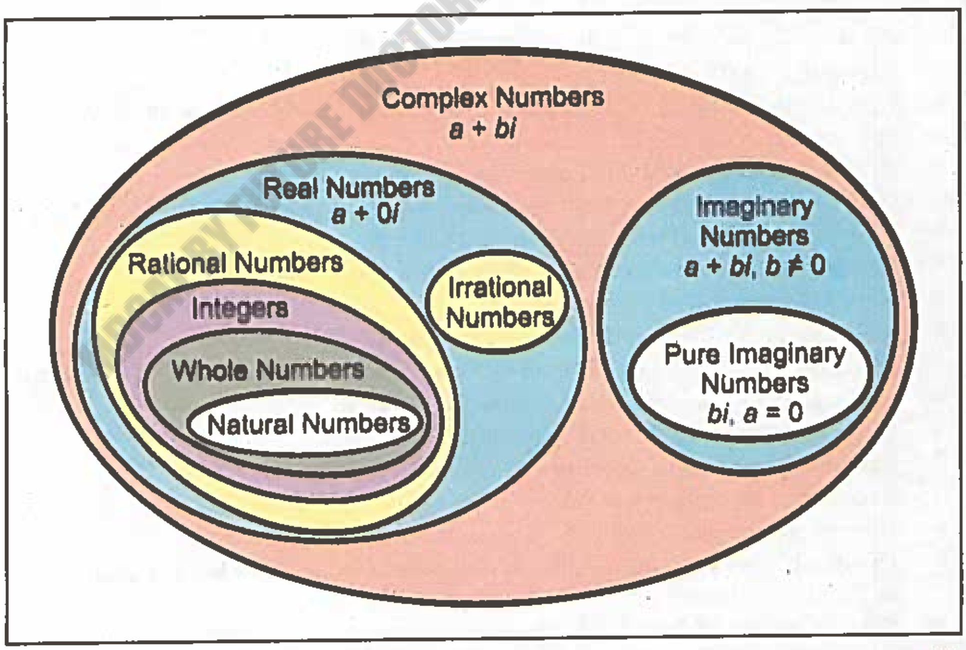
4. If $z_1 = 2 - i$, $z_2 = 1 + i$, find $\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right|$.

5. Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$.

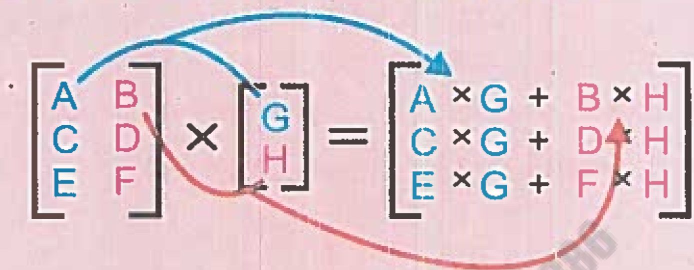
6. Find the conjugate of $\frac{1}{3+4i}$.

7. Find the multiplicative inverse of $z = \frac{3i+2}{3-2i}$.

8. Solve the quadratic equation $z + \frac{2}{z} = 2$.



UNIT 2 MATRICES AND DETERMINANTS



STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Recall the concept of
  - a matrix and its notation,
  - order of a matrix,
  - Equality of two matrices.
- Define row matrix, column matrix, square matrix, rectangular matrix, zero/null matrix, identity matrix, scalar matrix, diagonal matrix, upper and lower triangular matrix, transpose of a matrix, symmetric matrix and skew-symmetric matrix.
- Carryout scalar multiplication, addition/subtraction of matrices, multiplication of matrices with real and complex entries.
- Show that commutative property
  - holds under addition.
  - does not hold under multiplication, in general.
- Verify that $(AB)^t = B^t A^t$
- Describe determinant of a square matrix, minor and cofactor of an element of a matrix.
- Evaluate determinant of a square matrix using cofactors.
- Define singular and non-singular matrices.
- Know the adjoint of a square matrix.
- Use adjoint method to calculate inverse of a square matrix.
- Verify the result $(AB)^{-1} = B^{-1}A^{-1}$.
- State and prove the properties of determinants.
- Evaluate the determinant without expansion (i.e. using properties of determinants).
- Know the row and column operations on matrices.
- Define echelon and reduced echelon form of a matrix.
- Reduce a matrix to its echelon and reduced echelon form.
- Recognize the rank of a matrix.
- Use row operations to find the inverse and the rank of a matrix.
- Distinguish between homogeneous and non-homogeneous linear equations in 2 and 3 unknowns.
- Solve a system of three homogeneous linear equations in three unknowns.

- Define a consistent and inconsistent system of linear equations and demonstrate through examples.
- Solve a system of 3 by 3 non-homogeneous linear equations using:
  - matrix inversion method,
  - Gauss elimination method (echelon form),
  - Gauss-Jordan method (reduced echelon form),
  - Cramer's rule.

2.1 Introduction

The concept of matrices is a highly useful tool which is not only used in mathematics but also in all branches of science, engineering and the business world. Now-a-days matrices and matrix methods have widespread applications in the operation of high speed computers.

2.1.1 (a) Concept of a matrix and its notation

In previous class we have taken a simple example for the concept of a matrix. Here we take a bit more tricky example.

Suppose three colleges A,B,C take part in an inter-colleges debate competition, where any participant can speak in either of the four languages English, Urdu, Pashto or Hindko. College A consists of 3 participants in English, 2 in Urdu, 3 in Pashto and 1 in Hindko, College B consists of 2 participants in English, 3 in Urdu, 1 in Pashto and 2 in Hindko, College C consists of 4 participants in English, 2 in Urdu, 2 in Pashto and 1 in Hindko.

The information given in the above example, can be put in a compact way in a tabular form as follows:

Name of the School	Number of speakers (language wise)			
	English	Urdu	Pashto	Hindko
A	3	2	3	1
B	2	3	1	2
C	4	2	2	1

Now we write the data given in the above arrangement in a capital or small brackets without any top or left heading as shown. .

$$\begin{bmatrix} 3 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 4 & 2 & 2 & 1 \end{bmatrix} \quad or \quad \begin{pmatrix} 3 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 4 & 2 & 2 & 1 \end{pmatrix}$$

This array of numbers gives all the information needed which we call a matrix. Thus a **matrix** is a rectangular array of numbers enclosed in large square brackets or parenthesis. Unless otherwise specified, all numbers in a matrix array will be real. For example,

$$\text{rows}\ \begin{bmatrix} 3 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 4 & 2 & 2 & 1 \end{bmatrix} \quad \textit{or} \quad \text{rows}\ \begin{pmatrix} 3 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 4 & 2 & 2 & 1 \end{pmatrix}$$

columns *columns*

represents matrix. However, throughout we will use square brackets to denote matrices.

In the above matrix the horizontal lines of numbers are called **rows** and the vertical lines of numbers are called **columns**. Each number in the array is called an **element** or an **entry** of the matrix.

The above matrix has three rows and four columns.

We are now ready to give the general definition of a matrix as follows:

A matrix is a rectangular array of mn elements a_{ij}; $i = 1,2,3,\ldots,m$ $j=1,2,\ldots,n$ arranged in m rows and n columns. In writing down matrices, it is usual to denote the matrix by a capital single letter A (say) such that

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \vdots & \vdots \quad\ \vdots \\ a_{m1} & a_{m2} \ldots a_{mn} \end{bmatrix}$$

(b) Order of a matrix

The **order** of a matrix is given by the number of rows followed by the number of columns, if the matrix **A** has m rows and n columns, and so is said to be of order $m \times n$ (read as m by n matrix).

For simplicity and to convey the idea, the matrix **A** is an $m \times n$ matrix, unless otherwise specified.

In the matrix **A**, the ith row and the jth column are represented as follows:

jth column

$$A = \begin{bmatrix} a_{11} & a_{12} \dots & a_{1j} & \dots a_{1n} \\ a_{21} & a_{22} \dots & a_{2j} & \dots a_{2n} \\ \vdots & & & \\ a_{i1} & a_{i2} \dots & a_{ij} & \dots a_{in} \\ \vdots & & & \\ a_{m1} & a_{m2} \dots & a_{mj} & \dots a_{mn} \end{bmatrix} \leftarrow \text{ith row}$$

The elements of the ith row of A are $a_{i1}, a_{i2}, \dots, a_{ij}, \dots, a_{in}$ and the elements of the jth column of A are $a_{1j}, a_{2j}, \dots, a_{ij}, \dots, a_{mj}$. We see that the element a_{ij} occurs in the ith row and jth column of A. The elements in the ith row and jth column will usually be referred to as the (i,j)th element because of the two subscripts i and j.

We may also write the matrix A as

$A=[a_{ij}]_{m\times n}$ or $A=[a_{ij}]$; i=1,2,…,m; j=1,2,…,n, where a_{ij} is the (i,j)th elements of A.

Did You Know

A matrix is merely a table of numbers. Apart from being a convenient way of recording certain types of numerical values, it has no particular value in itself.

(c) Equality of two matrices

Two matrices $A=[a_{ij}]$ and $B = [b_{ij}]$ of the same order are said to be **equal** when their corresponding elements are equal i.e. $a_{ij} = b_{ij}$ for all *i* and *j* where *i*=1,2,…,m; *j*=1,2,…,n

For example, if

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \quad and \quad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \quad then \quad A = B.$$

2.1.2 Types of Matrices

(a) Row Matrix or Row Vector

A matrix with only one row i.e. a 1xn matrix of the form $[a_{11} \ a_{12} \dots a_{1n}]$ is called row matrix or a row vector. For example, $[-1 \quad -2 \quad -3]$ is a row matrix having three columns.

(b) Column matrix or Column vector

A matrix with only one column i.e. an m × 1 matrix of the form $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ is called a column matrix or a column vector.

For example, $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is a column matrix having four rows.

(c) Square matrix

If the number of rows and columns in a matrix are equal i.e. if m=n, then the matrix of order m×n is called a square matrix of order n or m.

For example, $A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}$

is a square matrix of order n and [a], $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 5 \\ 3 & 6 & 2 \end{bmatrix}$ are square matrices of order 1, 2 and 3 respectively.

The diagonal of the square matrix A containing the elements $a_{11}, a_{22}, \ldots, a_{nn}$ is called the **principal diagonal** of A. It is also termed as the **leading diagonal** or **main diagonal** of the matrix A.

(d) Rectangular matrix

If the number of rows and columns in a matrix A are not equal, i.e. if $m \neq n$, the matrix is called a rectangular matrix of order m × n.

For example, $\begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -1 & 2 \\ 3 & 5 & -4 & 3 \end{bmatrix}$

are rectangular matrices of order 2×3 and 3×4 respectively.

(e) Diagonal Matrix

A square matrix is called a diagonal matrix if all its non-diagonal elements are zero.

Thus, the square matrix $[a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.

For example, [2], $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

are diagonal matrices.

(f) Scalar matrix

A square matrix is called a scalar matrix, if its non-diagonal elements are zero and diagonal elements are equal.

Thus, the square matrix $[a_{ij}]$ is a scalar matrix if

$$a_{ij} = \begin{cases} k & for\, i = j \\ 0 & for\, i \neq j \end{cases}$$

For example $\begin{bmatrix} k & 0 & 0 & \ldots & 0 \\ 0 & k & 0 & \ldots & 0 \\ 0 & 0 & k & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & k \end{bmatrix}$ is a general scalar matrix of order n.

$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are scalar matrices of order 2 and 3 respectively.

(g) Unit matrix or Identity matrix

A square matrix is called a unit matrix if its non-diagonal elements are zero and diagonal elements are all equal to one (unity).

Thus, the square matrix $[a_{ij}]$ is a unit matrix if

$$a_{ij} = \begin{cases} 1 & for\ i = j \\ 0 & for\, i \neq j \end{cases}$$

Such a matrix is denoted by

$$I_n = \begin{bmatrix} 1 & 0 \ldots\ldots & 0 \\ 0 & 1 \ldots\ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \ldots\ldots & 1 \end{bmatrix},$$

We have unit matrices of different order such as

$$I_2=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad I_4=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\text{ and so on.}$$

(h) Zero matrix or Null matrix

A matrix all whose elements are zero is called a zero matrix or null matrix. If it has m rows and n columns, we denote it by $O_{m\times n}$ or simply by O if there is no ambiguity about its number of rows and number of columns.

Following are some examples of zero or null matrices:

$$[0]\ ,\ [0\ \ 0\ \ 0]\ ,\ \begin{bmatrix}0\\0\\0\end{bmatrix}\ ,\ \begin{bmatrix}0&0\\0&0\end{bmatrix}\ \textit{and}\ \begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}.$$

(i) Transpose of a matrix

Let $A=[a_{ij}]$ be an m×n matrix. The **transpose** of A denoted by A^t, is an n×m matrix obtained by interchanging rows and columns of A. Thus $A^t=[b_{ij}]$ where $b_{ij}=a_{ji}$ for $i=1,2,\ldots,n;\ j=1,2,\ldots,m$.

For example, if A is a 3×2 matrix given by $A=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\\a_{31}&a_{32}\end{bmatrix}$,

then its transpose A^t is a 2×3 matrix

$$A^t=\begin{bmatrix}a_{11}&a_{21}&a_{31}\\a_{12}&a_{22}&a_{32}\end{bmatrix}.$$

(j) Upper triangular matrix

A square matrix $A=[a_{ij}]_{m\times n}$ is said to be upper triangular matrix, if all the elements below the principal diagonal are zero that is $a_{ij}=0$ for all $i>j$.

For example, $\begin{bmatrix}2&3&1\\0&4&-2\\0&0&1\end{bmatrix}$ and $\begin{bmatrix}1&2&3&4\\0&-1&2&0\\0&0&6&3\\0&0&0&1\end{bmatrix}$ are upper triangular matrices.

(k) Lower triangular matrix

A square matrix $A=[a_{ij}]_{m\times n}$ is said to be lower triangular matrix, if all the elements above the principal diagonal are zero, that is $a_{ij}=0$ for all $i<j$.

For example, $\begin{bmatrix} 2 & 0 & 0 \\ -4 & 5 & 0 \\ 2 & 3 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & -2 & 3 & 0 \\ 1 & 0 & 3 & 2 \end{bmatrix}$ are lower triangular matrices.

(l) Triangular matrix

A square matrix A is called a triangular matrix, if it is either upper triangular or lower triangular.

Note

- It is obvious that diagonal matrices are both upper triangular and lower triangular.
- If A is triangular, then $|A|$ = product of diagonal elements.

For example,

$\begin{bmatrix} 2 & -3 & 4 \\ 0 & 4 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 4 & -2 & 3 & 0 \\ 1 & 0 & 3 & 2 \end{bmatrix}$ are triangular matrices.

The first matrix is upper triangular while the second is lower triangular.

(m) Symmetric matrix

A square matrix $A=[a_{ij}]$ of order n is said to be symmetric if $A^t = A$, that is, if $a_{ij}=a_{ji}$ for $i, j=1,2,\ldots,n$.

For example, the matrix $A=\begin{bmatrix} 2 & 3 & 6 \\ 3 & 1 & -5 \\ 6 & -5 & 4 \end{bmatrix}$ is symmetric, since $A^t=\begin{bmatrix} 2 & 3 & 6 \\ 3 & 1 & -5 \\ 6 & -5 & 4 \end{bmatrix}=A$

(n) Skew symmetric matrix

A square matrix $A=[a_{ij}]$ of order n is said to be skew symmetric (or anti symmetric), if $A^t=-A$, that is, if $a_{ij}=-a_{ji}$ for $i, j=1,2,\ldots,n$.

For elements on the principal diagonal, we have

$a_{ii}=-a_{ii} \Rightarrow 2a_{ii}=0 \Rightarrow a_{ii}=0$ for $i=1,2,\ldots,n$.

Thus the elements on principal diagonal of skew symmetric matrix are zero.

For example, the matrix $A=\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$ is skew symmetric,

$$\text{since } A' = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix} = (-1)\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix} = -A$$

2.2 Algebra of matrices

In this section various operations of addition, subtraction, multiplication etc on matrices are defined.

2.2.1. (a) Addition of matrices

If $A=[a_{ij}]$ and $B=[b_{ij}]$ are two matrices of the same order m×n, then their sum A+B is defined as a matrix $C=[c_{ij}]$ of the same order as **A** and **B** and whose elements are obtained by adding the corresponding elements of **A** and **B** together.

Symbolically, we write C=A+B whose elements $c_{ij} = a_{ij}+b_{ij}$ for $i=1,2,\ldots,m; j=1,2,\ldots,n$.

Did You Know

- If the sum of two matrices is defined, we say that the two matrices are conformable for addition.
- The sum of two matrices of different order is not defined that is, they are not conformable for addition.

For example, if $A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ *and* $B=\begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$, *then*

$$C = A+B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+3 & 2+4 & 3+5 \\ 0+1 & -1+2 & 2+3 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 8 \\ 1 & 1 & 5 \end{bmatrix}$$

(b) Subtraction of matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same order m × n, then subtraction of matrices A and B is obtained by subtracting the corresponding elements of A and B respectively. The difference of A and B (or the subtraction of B from A) is a matrix D = A–B whose elements are $d_{ij}= a_{ij} - b_{ij}$; $i= 1,2,\ldots\ldots,m$; $j= 1,2,\ldots\ldots,n$.

If $A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ then,

$$D = A - B = A + (-B) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -3 & -4 & -5 \\ -1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & 2-4 & 3-5 \\ 0-1 & -1-2 & 2-3 \end{bmatrix} = \begin{bmatrix} -2 & -2 & -2 \\ -1 & -3 & -1 \end{bmatrix}$$

(c) Scalar multiplication

If $A=[a_{ij}]$ is a matrix of order m×n and k is any scalar, then the scalar multiplication kA of the scalar k and matrix A is defined as a matrix each of whose element is the product of k and the corresponding elements of A i.e.

$kA = k[a_{ij}]=[ka_{ij}]$; $i=1,2,\ldots\ldots,m$; $j=1,2,\ldots\ldots,n$.

For example, if $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$

and k is any scalar, then

$$kA = k\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}k & 2k\\ 3k & 4k\end{bmatrix}.$$

Note

- Clearly kA is a matrix of the same order as the given matrix A.
- A + A = 2A, A + A + A = 3A and in general, if n is a positive integer, then $\underbrace{A + A + \ldots + A}_{n-times} = nA$

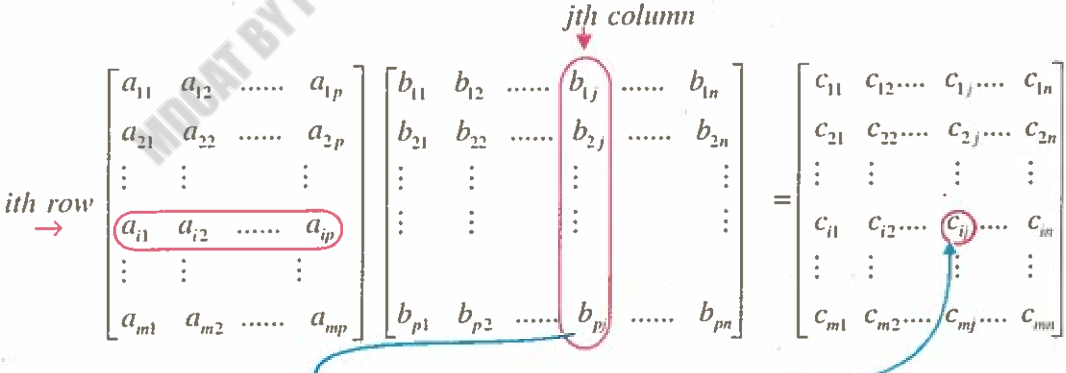
(d) Multiplication of matrices

Two matrices A and B are said to be conformable for multiplication giving the product AB, if the number of columns in **A** is equal to the number of rows in **B**.

Suppose **A** = $[a_{ij}]$ is matrix of order m×p and **B**= $[b_{ij}]$ is a matrix of order p×n. Then their product AB is a matrix $C=[c_{ij}]$ of order m×n with elements c_{ij} defined as the sum of the product of the corresponding elements of the ith row of **A** and the jth column of **B** i.e.

$$c_{ij} = a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots\ldots+a_{ip}b_{pj}=\sum_{k=1}^{p}a_{ik}b_{kj}$$

The following illustrates the expression for c_{ij}



$$a_{i1}b_{1j}+a_{i2}b_{2j} + \ldots\ldots\ldots + a_{ip}b_{pj}=c_{ij}$$

For example if $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix}$ are two matrices of order 2×3 and 3×2 respectively. Then the product C = AB is 2×2 matrix defined by

$$C = AB = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3\times1+1\times3+2\times2 & 3\times2+1\times1+2\times3 \\ 2\times1+1\times3+3\times2 & 2\times2+1\times1+3\times3 \end{bmatrix} = \begin{bmatrix} 10 & 13 \\ 11 & 14 \end{bmatrix}$$

The matrices A and B are also conformable for the product D=BA defined as

$$D = BA = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 8 \\ 11 & 4 & 9 \\ 12 & 5 & 13 \end{bmatrix}$$

C and D are matrices of order 2×2 and 3×3 respectively.

2.2.2 Commutative property

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 4 & -5 \end{bmatrix}$. Find AB and BA and show that $AB \neq BA$.

Here, A is a 2x3 matrix and B is a 3x2 matrix. So, AB exists and it is of order 2x2

We have, $AB = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 4 & -5 \end{bmatrix}$

$$= \begin{bmatrix} 2+2+12 & 3-4-15 \\ 6-2-4 & 9+4+5 \end{bmatrix} = \begin{bmatrix} 16 & -16 \\ 0 & 18 \end{bmatrix}$$

Again, B is a 3x2 matrix and A is a 2x3 matrix. So BA exists and it is of order 3x3

Now, $BA = \begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix}$

$$\Rightarrow BA = \begin{bmatrix} 2+9 & -4+6 & 6-3 \\ -1+6 & 2+4 & -3-2 \\ 4-15 & -8-10 & 12+5 \end{bmatrix} = \begin{bmatrix} 11 & 2 & 3 \\ 5 & 6 & -5 \\ -11 & -18 & 17 \end{bmatrix}$$

Clearly, AB $\neq$ BA

However, commutative property w.r.t. addition clearly holds if both matrices are conformable for addition and is explained below:

Commutative property w.r.t. addition, i.e., A + B = B + A.

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $B = \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix}$ be two 3 x 3 square matrices.

Then $A + B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} a+j & b+k & c+l \\ d+m & e+n & f+o \\ g+p & h+q & i+r \end{bmatrix}$ (1)

and $B + A = \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} + \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$= \begin{bmatrix} j+a & k+b & l+c \\ m+d & n+e & o+f \\ p+g & q+h & r+i \end{bmatrix} = \begin{bmatrix} a+j & b+k & c+l \\ d+m & e+n & f+o \\ g+p & h+q & i+r \end{bmatrix}$ (2)

Since addition is commutative in $\mathbb{R}$. From (1) and (2), we have A + B = B + A

Example 1: If $A = \begin{bmatrix} 3 & 2 \\ 4 & -1 \\ 6 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ -1 & 4 \\ 0 & 3 \end{bmatrix}$, then show that $(A+B)^t = A^t + B^t$.

Solution: Since

$A + B = \begin{bmatrix} 3 & 2 \\ 4 & -1 \\ 6 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ -1 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3+2 & 2+5 \\ 4-1 & -1+4 \\ 6+0 & 1+3 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 3 & 3 \\ 6 & 4 \end{bmatrix}$, So $(A+B)^t = \begin{bmatrix} 5 & 3 & 6 \\ 7 & 3 & 4 \end{bmatrix}$ (1)

Now $A^t = \begin{bmatrix} 3 & 4 & 6 \\ 2 & -1 & 1 \end{bmatrix}$, $B^t = \begin{bmatrix} 2 & -1 & 0 \\ 5 & 4 & 3 \end{bmatrix}$,

$$\therefore \quad A^t + B^t = \begin{bmatrix} 3 & 4 & 6 \\ 2 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 0 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 3+2 & 4-1 & 6+0 \\ 2+5 & -1+4 & 1+3 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 6 \\ 7 & 3 & 4 \end{bmatrix} \quad (2)$$

From (1) and (2), we have $(A+B)^t = A^t + B^t$.

2.2.3 Verification of $(AB)^t = B^t A^t$

Example 2: *If* $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}$, verify $(AB)^t = B^t A^t$

Solution: $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}$

$$\therefore AB = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} -2 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ -4 & -2 & -8 \\ -6 & -3 & -12 \end{bmatrix}$$

$$\Rightarrow (AB)^t = \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix} \quad \text{(i)}$$

Also, $B^t A^t = \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}^t \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}^t = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix} \quad \text{(ii)}$

From (i) and (ii), we observe that $(AB)^t = B^t A^t$

EXERCISE 2.1

1. **Express the following as a single matrix.**

(i) $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 5 \\ 0 & 2 & 4 \\ -7 & 5 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -5 & 7 \end{bmatrix}$

(iii) $\begin{bmatrix} 7 & 1 & 2 \\ 9 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}+2\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ (iv) $\left\{\begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}+\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\right\}\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

2. Let $A=\begin{bmatrix} 2 & -5 & 1 \\ 3 & 0 & -4 \end{bmatrix}$, $B=\begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & 5 \end{bmatrix}$ and $C=\begin{bmatrix} 0 & 1 & -2 \\ 0 & -1 & -1 \end{bmatrix}$.

Find 2A + 3B – 4C.

3. (i) if $A=\begin{bmatrix} x & y & z \end{bmatrix}$, $B=\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ and $C=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, verify that $(AB)C=A(BC)$

(ii) If $A=\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 1 & -3 \\ 0 & 4 & 2 \end{bmatrix}$ and $C=\begin{bmatrix} 3 & -1 & 5 \\ 2 & 1 & 0 \end{bmatrix}$, verify that:

(a) $A(B+C)=AB+AC$ (b) $A(B-C)=AB-AC$

4. Let $A=\begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 4 \\ 4 & 4 & 1 \end{bmatrix}$, show that $\frac{1}{3}A^2-2A-9I=O$.

5. Matrix $A=\begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ is given to be symmetric, find values of a and b.

6. **Solve the following matrix equations for X.**

(i) $X-3A=2B$, if $A=\begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 4 \end{bmatrix}$

(ii) $2(X-A)=B$, if $A=\begin{bmatrix} 1 & 2 & 2 \\ 3 & -1 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 4 & 6 & 2 \\ 0 & -4 & 2 \end{bmatrix}$

7. *If* $A=\begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \\ 0 & -2 & 1 & 6 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -1 & 4 \\ 3 & 1 & 2 & -1 \end{bmatrix}$,

then show that $(A+B)^t=A^t+B^t$.

8. Let $A=\begin{bmatrix}1 & 2 & 0\\3 & -1 & 4\end{bmatrix}$. Show that

(i) $(A')'=A$ (ii) $AA'\neq A'A$

9. Verify that $(AB)'=B'A'$ if

(i) $A=\begin{bmatrix}2 & -1 & 3\\1 & 0 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 2\\2 & 2\\3 & 0\end{bmatrix}$ (ii) $A=\begin{bmatrix}1 & 2 & 0\\-1 & 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}1 & 1\\2 & 3\\1 & -2\end{bmatrix}$

10. Let $A=\begin{bmatrix}1 & -3 & 4\\-3 & 2 & -5\\4 & -5 & 0\end{bmatrix}$ and $B=\begin{bmatrix}5 & 6 & 7\\6 & -8 & 3\\7 & 3 & 1\end{bmatrix}$

Verify that A and B are symmetric. Also verify that A + B is symmetric.

11. Let $A=\begin{bmatrix}0 & 1 & -2\\-1 & 0 & 3\\2 & -3 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & -6 & 11\\6 & 0 & -7\\-11 & 7 & 0\end{bmatrix}$

Verify that A + B is skew-symmetric.

12. If $A=\begin{bmatrix}3 & 2 & 1\\4 & 5 & 6\\-2 & 3 & 4\end{bmatrix}$, then verify that

(i) $A+A^t$ is symmetric (ii) $A-A^t$ is skew-symmetric.

13. If A is a square matrix of order 3, then show that:

(i) $A+A^t$ is symmetric (ii) $A-A^t$ is skew-symmetric.

2.3. Determinants

Consider a square matrix A of order n given by

$$A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\a_{21} & a_{22} & \cdots & a_{2n}\\\vdots & \vdots & & \vdots\\a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix} \quad (I)$$

The associated determinant of A is denoted by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad (2)$$

Some determinants of higher order can be evaluated only after much tedious calculations. The more calculation is involved, the greater the chance of error. Our aim in this section is to describe a procedure for evaluating the determinants of order $n \geq 3$. However, this procedure will be greatly simplified by the introduction of the following.

2.3.1. Minor and Cofactor of an element of a matrix or its determinants

(i) Minor of an Element Let A be a square matrix of order n (as defined in (1) above). The minor of the element a_{ij} of A, denoted by $\mathbf{M_{ij}}$, is the determinant of $(n-1)\times(n-1)$ matrix obtained by crossing out the ith row and jth column of A (or |A|).

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ obtained as $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,

minor of $a_{23} = M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$ obtained as $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and so on.

Remember

From the formula $A_{ij} = (-1)^{i+j} M_{ij}$ it is clear that if the sum i+j is an even integer, then the cofactor equals the minor. On the other hand, if the sum i+j is odd, the cofactor is equal to the negative of the minor. The signs accompanying the minors may be best remembered by the rule of alternating signs with +'s on the main diagonals.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Example 3: Let $A=\begin{bmatrix}1 & 2 & 3\\ 6 & 5 & 4\\ 7 & 8 & 9\end{bmatrix}$. Find the minors M_{11}, M_{12}, M_{13} and M_{22} of the matrix A.

Solution: We have

$$M_{11}=\begin{vmatrix}5 & 4\\ 8 & 9\end{vmatrix}=45-32=13, M_{12}=\begin{vmatrix}6 & 4\\ 7 & 9\end{vmatrix}=54-28=26,$$

$$M_{13}=\begin{vmatrix}6 & 5\\ 7 & 8\end{vmatrix}=48-35=13, M_{22}=\begin{vmatrix}1 & 3\\ 7 & 9\end{vmatrix}=9-21=-12.$$

(ii) Cofactor of an element

Let A be a square matrix of order n. The cofactor of the element a_{ij}, denoted by A_{ij}, is defined by $A_{ij}=(-1)^{i+j}M_{ij}$, where M_{ij} is the minor of a_{ij}.

Thus if $A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$, then

cofactor of $a_{11}=A_{11}=(-1)^{1+1}M_{11}=(-1)^2\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}$

$$=1\times(a_{22}a_{33}-a_{23}a_{32})$$

$$=a_{22}a_{33}-a_{23}a_{32}$$

cofactor of $a_{23}=A_{23}=(-1)^{2+3}M_{23}=(-1)^5\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix}$

$$=-1\times(a_{11}a_{32}-a_{12}a_{31})$$

$=-(a_{11}a_{32}-a_{12}a_{31})$ and so on.

Example 4: Let $A=\begin{bmatrix}1 & -2 & 5\\ 3 & 0 & -1\\ 5 & 2 & 0\end{bmatrix}$. Find the cofactor A_{13} and A_{21}.

Solution: We have $A_{13}=(-1)^{1+3}M_{13}=(-1)^4\begin{vmatrix}3 & 0\\ 5 & 2\end{vmatrix}=1\times(3\times2-0\times5)=6,$

and $A_{21}=(-1)^{2+1}M_{21}=(-1)^3\begin{vmatrix}-2 & 5\\ 2 & 0\end{vmatrix}=-1\times(-2\times0-5\times2)=10.$

2.3.2 Determinant of a square matrix of order $n \geq 3$

Let A be a square matrix of order $n(\geq 3)$ given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1j} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2j} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots\cdots & a_{ij} & \cdots\cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nj} & \cdots\cdots & a_{nn} \end{bmatrix} \quad (1)$$

The determinant |A| of the matrix A is defined to be the sum of the products of each element of row (or column) and its cofactor, that is

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots\ldots + a_{in}A_{in}; i = 1, 2\ldots\ldots, n \quad (2)$$

or $$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots\ldots + a_{nj}A_{nj}; j = 1, 2\ldots\ldots, n \quad (3)$$

If we put $i=1$ in (2), we get

$|A| = a_{11}A_{11} + a_{12}A_{12} + \ldots\ldots + a_{1n}A_{1n}$. This is called the expansion of |A| by first row (or w.r.t. first row).

Similarly, if we put j=1 in (3), we get

$|A| = a_{11}A_{11} + a_{21}A_{21} + \ldots\ldots + a_{n1}A_{n1}$. This is called the expansion of |A| by first column and so on. Thus, if A is a square matrix of order 3, that is

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then by (2) and (3), we have

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3}; \qquad i = 1, 2, 3 \quad (2')$$

or $$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j}; \quad j = 1, 2, 3 \quad (3')$$

For example, if $i=2$, then by ($2'$), we have

$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$. This can be written as

$$|A| = a_{21}(-1)^{2+1}M_{21} + a_{22}(-1)^{2+2}M_{22} + a_{23}(-1)^{2+3}M_{23}.$$

$$= -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{23}(a_{11}a_{32} - a_{12}a_{31})$$

$$= -a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} + a_{22}a_{11}a_{33} - a_{22}a_{13}a_{31} - a_{23}a_{11}a_{32} + a_{23}a_{12}a_{31}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \quad (4)$$

Similarly, we can find |A| for other values of i and j.

The expansion of |A| in **(4)** can also be remembered by the following procedure.

Rewrite the first two columns of the matrix A after the third column and use the following diagram, if A is a 3x3 matrix.

(5)

The arrows pointing downward represent the three products having a positive sign and the arrows pointing upward represent the three products having a negative sign.

Example 5: If $A = \begin{bmatrix} 3 & -1 & 2 \\ 3 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, then find $|A|$.

Solution: $|A| = \begin{vmatrix} 3 & -1 & 2 \\ 3 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix}$

We expand the determinant by using the elements of the first row, we have

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \quad (1)$$

But $A_{11} = (-1)^{1+1}M_{11} = M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$

$A_{12} = (-1)^{1+2}M_{12} = -M_{12} = -\begin{vmatrix} 3 & 0 \\ 1 & -1 \end{vmatrix}$

$A_{13} = (-1)^{1+3}M_{13} = M_{13} = \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix}$

Putting these values in **(1)**, we obtain

$$|A| = (3)\begin{vmatrix}1 & 0\\0 & -1\end{vmatrix} - (-1)\begin{vmatrix}3 & 0\\1 & -1\end{vmatrix} + (2)\begin{vmatrix}3 & 1\\1 & 0\end{vmatrix}$$

$$= (3)[(1\times -1 - 0\times 0)] - (-1)[(3\times -1 - 0\times 1)] + (2)[(3\times 0 - 1\times 1)]$$

$$= -3 - 3 - 2 = -8$$

We now expand the same determinant by using elements of the third column, that is $|A| = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$ (2)

Now

$$A_{13} = (-1)^{1+3}M_{13} = M_{13} = \begin{vmatrix}3 & 1\\1 & 0\end{vmatrix}$$

$$A_{23} = (-1)^{2+3}M_{23} = -M_{23} = -\begin{vmatrix}3 & -1\\1 & 0\end{vmatrix}$$

$$A_{33} = (-1)^{3+3}M_{33} = M_{33} = \begin{vmatrix}3 & -1\\3 & 1\end{vmatrix}$$

> **Note**
>
> We get the same result, no matter which row or column is used to expand a 3×3 determinant.
> The determinant of the square matrix A of order 3 in the above example can also be evaluated by the two simple methods given in (4) and (5).

Putting in (2), we get

$$|A| = (2)\begin{vmatrix}3 & 1\\1 & 0\end{vmatrix} - 0\begin{vmatrix}3 & -1\\1 & 0\end{vmatrix} + (-1)\begin{vmatrix}3 & -1\\3 & 1\end{vmatrix}$$

$$= (2)(3\times 0 - 1\times 1) - 0(3\times 0 + 1\times 1) - 1(3\times 1 + 1\times 3) = -2 - 0 - 6 = -8.$$

2.3.3 Singular matrix and non-singular matrix

A square matrix A is called a singular matrix if its determinant is zero, i.e. $|A| = 0$, otherwise, it is a non-singular matrix.

If $A = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$, then $|A| = \begin{vmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{vmatrix} = 1(45-48) - 2(36-42) + 3(32-35) = 0$

Therefore, A is a singular matrix

2.3.4 Adjoint of a square matrix

Let A be a square matrix of order n. Let $\mathcal{A}$ denote the matrix obtained by replacing each element of A by its corresponding cofactor. Then $\mathcal{A}^t$ is called the adjoint of A and is usually denoted by adj A i.e. adj A = $\mathcal{A}^t$

Thus, *if* $A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$, then $\mathcal{A} = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A_{33}\end{bmatrix}$

and so adj A = $\mathcal{A}^t = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

For example, if $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$, then $\mathcal{A} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & -1 & -1 \\ 2 & 2 & -2 \end{bmatrix}$

and so adj A $= \mathcal{A}^t = \begin{bmatrix} 3 & -1 & 2 \\ -1 & -1 & 2 \\ -1 & -1 & -2 \end{bmatrix}$.

2.3.5 Use adjoint method to calculate inverse of a square matrix

Let A be a square matrix of order n. If there exists a square matrix B of order n such that AB = BA = I_n where I_n is the multiplicative identity matrix of order n, then B is called the **multiplicative inverse** of A and is denoted by A^{-1}.

Thus $AA^{-1} = A^{-1}A = I_n$.

It may be noted that inverse of a square matrix, if it exists, is unique. Moreover, if A is a non-singular square matrix of order n, then $A^{-1} = \frac{1}{|A|}$ adj A.

Example 6: *Let* $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ -1 & 2 & 0 \end{bmatrix}$. *Find* A^{-1}.

Solution: Since $A^{-1} = \frac{1}{|A|} adj\, A$, we need to find adj A and | A |.

First we find co-factor of every element of A.

$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix} = 1\cdot(0+4) = 4,$ $\quad A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & -2 \\ -1 & 0 \end{vmatrix} = -1\cdot(0-2) = 2$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} = 1\cdot(0+1) = 1,$ $\quad A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 2 & 0 \end{vmatrix} = -1\cdot(0-2) = 2$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 1\\ -1 & 0\end{vmatrix}=1\cdot(0+1)=1, \qquad A_{23}=(-1)^{2+3}\begin{vmatrix}1 & -2\\ -1 & 2\end{vmatrix}=-1\cdot(2-2)=0$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}-2 & 1\\ 1 & -2\end{vmatrix}=1\cdot(4-1)=3, \qquad A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 1\\ 0 & -2\end{vmatrix}=-1\cdot(-2-0)=2$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1 & -2\\ 0 & 1\end{vmatrix}=1\cdot(1+0)=1$$

$$So\ \ adj\,A=\begin{bmatrix}4 & 2 & 3\\ 2 & 1 & 2\\ 1 & 0 & 1\end{bmatrix}$$

Next we find |A|.

Since $|A|=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}$

$=1.(4)-2(2)+1(1)$

$=4-4+1=1\neq 0.$

Thus $A^{-1}=\dfrac{1}{|A|}adj\,A=1\cdot\begin{bmatrix}4 & 2 & 3\\ 2 & 1 & 2\\ 1 & 0 & 1\end{bmatrix}=\begin{bmatrix}4 & 2 & 3\\ 2 & 1 & 2\\ 1 & 0 & 1\end{bmatrix}$

2.4 Properties of determinants

We shall state some of the useful properties of determinants which simplify the evaluation of determinants.

Property 1. If every element in a row or column of a square matrix A is zero, then |A|=0.

If $A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ and every element in the first row is zero,

then

$$A=\begin{bmatrix}0 & 0 & 0\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

Now $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 0A_{11} + 0A_{12} + 0A_{13} = 0.$

We get the same result if every element of any other row or column is zero.

Property 2. If all elements of the corresponding rows and columns of a square matrix A are interchanged, then the determinant of the resulting matrix is equal to |A|. That is, the determinant of a square matrix and its transpose are always same.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$, then

|B| = |A|. Proof is left as an exercise.

Property 3. If any two rows or two columns in a square matrix A are interchanged, then the determinant of the resulting matrix is –|A|. In other words, both the determinants are additive inverses of each other.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the matrix obtained by interchanging the first and second row of A, then

$$|B| = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{21}(a_{12}a_{33} - a_{13}a_{32}) - a_{22}(a_{11}a_{33} - a_{13}a_{31}) + a_{23}(a_{11}a_{32} - a_{21}a_{31})$$

$$= a_{21}a_{12}a_{33} - a_{21}a_{13}a_{32} - a_{22}a_{11}a_{33} + a_{22}a_{13}a_{31} + a_{23}a_{11}a_{32} - a_{23}a_{12}a_{31}$$

$$= -(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{21}a_{31})$$

$$= -|A|.$$

Property 4. If a square matrix A has two identical rows or two identical columns, then |A|=0

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a matrix obtained by interchanging the first and second rows of A. Then by property (3), |B| = –|A|. But the first and second rows of A are identical, mean A=B and so

|A|=|B|. Hence |A| = −|A| or 2|A| = 0 or |A| = 0. The same result is obtained if any two columns are identical.

Property 5. If every element of a row or column of a square matrix A is multiplied by the real number k, then the determinant of the resulting matrix is k|A|.

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B=\begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the matrix obtained by multiplying first row of A by k. Then

$$|B|=\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{aligned} &= ka_{11}A_{11}+ka_{12}A_{12}+ka_{13}A_{13} \\ &= k(a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}) \\ &= k|A|. \end{aligned}$$

A similar result is obtained if any other row or column is multiplied by k.

Property 6. If every element of a row or column of a square matrix A is the sum of two terms, then its determinant can be written as the sum of two determinants.

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then, $|A|=\begin{bmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{bmatrix}$.

Expanding by the first column, we have

$$|A|=(a_{11}+b_{11})A_{11}+(a_{21}+b_{21})A_{21}+(a_{31}+b_{31})A_{31}$$
$$=(a_{11}A_{11}+a_{21}A_{21}+a_{31}A_{31})+(b_{11}A_{11}+b_{21}A_{21}+b_{31}A_{31})$$
$$=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

Property 7. If every element of any row or column of a square matrix is multiplied by a real number k and the resulting product is added to the corresponding elements of another row or column of the matrix, then the determinant of the resulting matrix is equal to the determinant of the original matrix.

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $B=\begin{bmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ a_{21}+ka_{22} & a_{22} & a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{bmatrix}$ is the matrix obtained by multiplying every element of the second column of A and then adding to the corresponding element of the first column of A, then

$$|B|=\begin{vmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ a_{21}+ka_{22} & a_{22} & a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} ka_{12} & a_{12} & a_{13} \\ ka_{22} & a_{22} & a_{23} \\ ka_{32} & a_{32} & a_{33} \end{vmatrix} \quad \text{by property (6)}$$

$$=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}+k\begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix} \quad \text{by property (5)}$$

$$=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}+k(0) \quad \text{by property (4)}$$

$$=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=|A|.$$

EXERCISE 2.2

1. *If* $A=\begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$, then find $A_{11}, A_{21}, A_{23}, A_{31}, A_{32}, A_{33}$. Also find $|A|$.

2. Without evaluating state the reasons for the following equalities.

(i) $\begin{vmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 0 \end{vmatrix}=0$

(ii) $\begin{vmatrix} 1 & 2 & 3 \\ -8 & 4 & -12 \\ 2 & -1 & 3 \end{vmatrix}=0$

(iii) $\begin{vmatrix} 1 & 3 & -2 \\ 3 & -1 & 1 \\ 2 & 1 & 4 \end{vmatrix}=\begin{vmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ -2 & 1 & 4 \end{vmatrix}$

(iv) $\begin{vmatrix} 3 & 2 & 0 \\ 1 & 1 & -3 \\ 2 & 4 & -6 \end{vmatrix}=-3\begin{vmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{vmatrix}$

(v) $\begin{vmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 1 & -1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$ (vi) $\begin{vmatrix} 2 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 5 & 5 & 6 \\ 1 & 2 & 2 \end{vmatrix}$

3. Let A be a square matrix of order 3, then verify $|A'| = |A|$.

4. Evaluate the following determinants.

(i) $\begin{vmatrix} 0 & 1 & 3 \\ -1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & 4 & -2 \\ 2 & 4 & -6 \\ -4 & 2 & 0 \end{vmatrix}$

(iii) $\begin{vmatrix} 3 & 1 & 2 \\ 6 & -5 & 4 \\ -9 & 8 & -7 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & 1 & -3 \\ 1 & 1 & 0 \\ -2 & 3 & 4 \end{vmatrix}$

5. Show that

(i) $\begin{vmatrix} a & b & c \\ l & m & n \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & l & x \\ b & m & y \\ c & n & z \end{vmatrix}$ (ii) $\begin{vmatrix} a & b & c \\ 1-3a & 2-3b & 3-3c \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix} = 0$ (iv) $\begin{vmatrix} bc & ca & ab \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

6. Prove that

(i) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-a \end{vmatrix} = 0.$

(ii) $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(iii) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(iv) $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

(v) $\begin{vmatrix} bc & a^3 & \frac{1}{a} \\ ca & b^3 & \frac{1}{b} \\ ab & c^3 & \frac{1}{c} \end{vmatrix} = 0, \quad a \neq 0, b \neq 0, c \neq 0$

7. Evaluate (i) $\begin{vmatrix} 3860 & 3861 \\ 3862 & 3863 \end{vmatrix}$ (ii) $\begin{vmatrix} 81 & 82 & 83 \\ 84 & 85 & 86 \\ 87 & 88 & 89 \end{vmatrix}$

8. Prove that $\begin{vmatrix} 1+x & y & z \\ x & 1+y & z \\ x & y & 1+z \end{vmatrix} = 1+x+y+z$

9. Prove that $\begin{vmatrix} x & p & q \\ p & x & q \\ p & q & x \end{vmatrix} = (x-p)(x-q)(x+p+q)$

10. Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

11. Identify singular and non-singular matrices.

(i) $\begin{bmatrix} 7 & 1 & 3 \\ 6 & 2 & -2 \\ 5 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 1 \\ 3 & -2 & 1 \\ -2 & -3 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 2 & -3 \\ 3 & 6 & -3 \\ -1 & 0 & 1 \end{bmatrix}$

12. Find the value of λ if A is singular matrix. Where $A=\begin{bmatrix}-\lambda & 1 & 0\\ 1 & -\lambda & 1\\ 0 & 1 & -\lambda\end{bmatrix}$

13. Solve for x

(i) $\begin{vmatrix}x & 2 & 3\\ 0 & -1 & 1\\ 0 & 4 & 5\end{vmatrix}=9$ (ii) $\begin{vmatrix}-1 & 0 & 1\\ x^2 & 1 & x\\ 2 & 3 & 4\end{vmatrix}=-6$ (iii) $\begin{vmatrix}x+2 & 3 & 4\\ 2 & x+3 & 4\\ 2 & 3 & x+4\end{vmatrix}=0$

14. Show that if inverse of a square matrix exists, then it is unique.

15. *Let* $A=\begin{bmatrix}0 & 2 & 2\\ -1 & 3 & 2\\ 1 & 0 & 5\end{bmatrix}$. *Find* A^{-1}

16. Let $A=\begin{bmatrix}3 & -1\\ 4 & 2\end{bmatrix}$. Show that $|A^{-1}|=\dfrac{1}{|A|}$

17. Verify that $(AB)^{-1}=B^{-1}A^{-1}$ if $A=\begin{bmatrix}2 & 3\\ 1 & 0\end{bmatrix}$, $B=\begin{bmatrix}-1 & 1\\ 2 & 3\end{bmatrix}$

18. If A and B are non-singular matrices, then show that

(i) $(A^{-1})^{-1}=A$ (i) $(AB)^{-1}=B^{-1}A^{-1}$

19. Let $A=\begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix}$. Verify that $(A^{-1})'=(A')^{-1}$

2.5 Row and column operations

2.5.1 (a) Row operations on matrices

The following three operations performed on matrices are called (elementary) row operations:

(i) Interchanging of any two rows.

(ii) Multiplication of a row by any non-zero scalar.

(iii) Addition of any multiple of one row to another row.

Notations: We use the following notations to express the elementary row operations (i), (ii) and (iii):

- Interchanging of row R_i and R_j is represented by $R_i \leftrightarrow R_j$.

- Multiplication of a row R_i by a non-zero scalar k is denoted by kR_i.
- Adding k times R_i to R_j is expressed as R_j+kR_i.

(b) Column operations on matrices

The following three operations performed on matrices are called elementary column operations:

(i) Interchanging of any two columns i.e. $C_i \leftrightarrow C_j$.

(ii) Multiplication of a column by any non-zero scalar k i.e. kC_i

(iii) Addition of any multiple of one column to another column i.e. $C_i + kC_j$, where C_i, C_j are any two columns and k is any non-zero scalar.

If A is an m×n matrix, then an m×n matrix B obtained from A by performing a finite number of elementary row operations on A is called row equivalent to A. Symbolically, we write $B \overset{R}{\sim} A$ to denote B is row equivalent to A.

Similarly, we can define a column equivalent matrix that is replacing the word "row" by "column" in the above definition. We write $B \overset{C}{\sim} A$ to denote B is column equivalent to A.

Example 7: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ -1 & -4 \end{bmatrix}$. Perform the following elementary row and column operations on A.

(i) $R_3 \leftrightarrow R_1$ (ii) $C_1 \leftrightarrow C_2$ (iii) $R_2 + 2R_1$ (iv) $C_2 - C_1$ (v) $R_1 - 4R_3$.

Solution: $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ -1 & -4 \end{bmatrix}$

(i) $R_3 \leftrightarrow R_1 : \begin{bmatrix} -1 & -4 \\ 3 & 5 \\ 1 & 2 \end{bmatrix}$ (ii) $C_1 \leftrightarrow C_2 : \begin{bmatrix} 2 & 1 \\ 5 & 3 \\ -4 & -1 \end{bmatrix}$

(iii) $R_2 + 2R_1 : \begin{bmatrix} 1 & 2 \\ 3+2(1) & 5+2(2) \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 9 \\ -1 & -4 \end{bmatrix}$

(iv) $C_2 - C_1 : \begin{bmatrix} 1 & 2+(-1) \\ 3 & 5+(-3) \\ -1 & -4+(-(-1)) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ -1 & -3 \end{bmatrix}$

(v) $R_1 - 4R_3 : \begin{bmatrix} 1+(-4(-1)) & 2+(-4(-4)) \\ 3 & 5 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} 5 & 18 \\ 3 & 5 \\ -1 & -4 \end{bmatrix}$

2.5.2 Echelon and reduced echelon form of a matrix

(a) Echelon form of a matrix

An m×n matrix A is said to be in (row) echelon form (or an echelon matrix) if it satisfies the following properties.

(i) In each successive non-zero row, the number of zeros before the first non-zero entry of a row increases row by row,

(ii) Every non-zero row in A precedes every zero row (if there is any).

For example, the matrices $\begin{bmatrix} 2 & 3 & -4 & 1 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ and $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ are in echelon form, but the matrix $\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is not in echelon form.

(b) Reduced echelon form of a matrix

An m×n matrix A is said to be in reduced (row) echelon form (or reduced echelon matrix) if it satisfies the following properties.

(i) It is in (row) echelon form,

(ii) The first non-zero entry in R_i lies in C_j is 1 and all other entries of C_j are zero.

For example, the matrices $\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are in (row) reduced echelon form but $\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are not in (row) reduced echelon form.

2.5.3 Reduce a matrix to its echelon and reduced echelon form

Example 8: Reduce $A = \begin{bmatrix} 2 & 3 & -4 \\ 3 & 1 & -1 \\ 1 & -2 & -5 \end{bmatrix}$ to echelon form and then to reduced echelon form.

Solution: $\begin{bmatrix} 2 & 3 & -4 \\ 3 & 1 & -1 \\ 1 & -2 & -5 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & -2 & -5 \\ 3 & 1 & -1 \\ 2 & 3 & -4 \end{bmatrix}$ by $R_1 \leftrightarrow R_3$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 7 & 14 \\ 2 & 3 & -4 \end{bmatrix}$ by $R_2 - 3R_1$ $\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 7 & 14 \\ 0 & 7 & 6 \end{bmatrix}$ by $R_3 - 2R_1$

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 2 \\ 0 & 7 & 6 \end{bmatrix}$ by $\frac{1}{7} R_2$ $\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & -8 \end{bmatrix}$ by $R_3 - 7R_2$ (1)

$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ by $-\frac{1}{8} R_3$ $\underset{\sim}{R} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ by $R_1 + 2R_2$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by $R_1 + R_3$ and $R_2 - 2R_3$ (2)

The matrices in (1) and (2) are in echelon form and reduced echelon form of the given matrix A respectively.

2.5.4 Rank of a Matrix

Let A be a non-zero matrix. The rank of the matrix A is the number of non-zero rows in its (row) echelon form.

2.5.5 Using elementary row operation (ERO) to find the inverse and the rank of a matrix

(a) To find inverse of a matrix

Let A be a non-singular matrix. If we perform successive elementary row operations on the matrix [A | I], which reduce A to I and I to the resulting matrix B i.e. if [A | I] is reduced to [I | B], then B is the inverse of A written as A^{-1}.
Similarly, if we perform successive elementary column operation on the matrix [A | I], which reduces A to I and I to the resulting matrix C, then C is the inverse of A written as A^{-1}.

Example 9: Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 4 & 2 \\ -1 & 2 & -2 \end{bmatrix}$

Solution: Since $\begin{vmatrix} 2 & 3 & 1 \\ 5 & 4 & 2 \\ -1 & 2 & -2 \end{vmatrix}$

$$= 2\begin{vmatrix} 4 & 2 \\ 2 & -2 \end{vmatrix} - 3\begin{vmatrix} 5 & 2 \\ -1 & -2 \end{vmatrix} + 1\begin{vmatrix} 5 & 4 \\ -1 & 2 \end{vmatrix} \text{ (expanding by first row)}$$

$$= 2(-8-4) - 3(-10+2) + (10+4) = -24 + 24 + 14 = 14 \neq 0.$$

So A is non-singular and A^{-1} exists.

$$\text{Now} \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 5 & 4 & 2 & 0 & 1 & 0 \\ -1 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc|ccc} -1 & 2 & -2 & 0 & 0 & 1 \\ 5 & 4 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array}\right] \text{ by } R_1 \leftrightarrow R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 0 & 0 & -1 \\ 5 & 4 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array}\right] \text{ by} (-1)R_1$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 0 & 0 & -1 \\ 0 & 14 & -8 & 0 & 1 & 5 \\ 0 & 7 & 23 & 1 & 0 & 2 \end{array}\right] \text{ by } R_2 - 5R_1 \text{ and } R_3 - 2R_1$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 0 & 0 & -1 \\ 0 & 1 & \frac{-4}{7} & 0 & \frac{1}{14} & \frac{5}{14} \\ 0 & 7 & -3 & -1 & 0 & -2 \end{array}\right] \text{ by } \frac{1}{14}R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{-6}{7} & 0 & \frac{-1}{7} & \frac{-2}{7} \\ 0 & 1 & \frac{-4}{7} & 0 & \frac{1}{14} & \frac{5}{14} \\ 0 & 0 & -3 & -1 & \frac{-1}{6} & \frac{-7}{6} \end{array}\right] \text{ by } R_1-2R_2 \text{ and } R_3-7R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-6}{7} & \frac{-4}{7} & \frac{-29}{7} \\ 0 & 1 & 0 & \frac{2}{3} & \frac{-5}{8} & \frac{-1}{18} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{-1}{18} & \frac{7}{18} \end{array}\right] \text{ by } R_1+\frac{6}{7}R_3 \text{ and } R_2+\frac{4}{7}R_3$$

$$\text{Thus } A^{-1} = \begin{bmatrix} \frac{-6}{7} & \frac{-4}{7} & \frac{-29}{7} \\ \frac{2}{3} & \frac{-5}{8} & \frac{-1}{18} \\ \frac{1}{3} & \frac{-1}{18} & \frac{7}{18} \end{bmatrix}$$

(b) To find rank of a matrix

Example 10: Find the rank of $A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$

Solution: $A=\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ by $R_1 \leftrightarrow R_2$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \text{ by } R_2-4R_1 \text{ and } R_3-7R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ by } R_3-2R_2$$

The last matrix is the echelon form of A having 2 non-zero rows. Hence the rank of A is 2.

EXERCISE 2.3

1. Reduce each of the following matrices to the indicated form

(i) $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 3 & 4 & -5 \end{bmatrix}$ Echelon form (ii) $\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$ Reduced echelon form

(iii) $\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 2 \\ 4 & 1 & 7 \end{bmatrix}$ Reduced echelon form (iv) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$ Echelon form

2. Find the inverses of the following matrices by using elementary row operation.

(i) $\begin{bmatrix} 4 & -2 & 5 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -1 & 6 \\ 1 & 3 & 4 \\ -1 & 5 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

3. Find the ranks of each of the following matrices.

(i) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 & -4 \\ 0 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix}$

4. Find

RANK OF MATRIX

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

4×4 matrix

2.6 System of linear equations

2.6.1 Homogeneous and non-homogeneous linear equations

Consider the equation $ax + by = k$ (1)

where $a \neq 0$, $b \neq 0$ and $k \neq 0$. The equation (1) is called a non-homogeneous linear equation in two variables (or unknowns) x and y.

Now consider the following two non-homogeneous linear equations in two variables x and y.

$$\left.\begin{aligned} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{aligned}\right\} \quad (2)$$

These two equations together form a system of **non-homogeneous linear equations** in two variables x and y.

If we take k = 0 in equations (1), then it takes the form $ax + by = 0$ (3)

and is called a homogeneous linear equation in two variables x and y. If we take $k_1 = k_2 = 0$ in (2), then

$$\left.\begin{aligned} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{aligned}\right\} \quad (4)$$

is called a system of **homogeneous linear equations** in the variables x and y.

Similarly, the following equation,

$ax + by + cz = k$, where $a \neq 0$, $b \neq 0$, $c \neq 0$, and $k \neq 0$ (5)

is called a **non-homogeneous linear equation** in three variables x, y and z and the following three non-homogeneous linear equations in three variables x, y and z.

$$\left.\begin{aligned} a_1x + b_1y + c_1z = k_1 \\ a_2x + b_2y + c_2z = k_2 \\ a_3x + b_3y + c_3z = k_3 \end{aligned}\right\} \quad (6)$$

together form a system of non-homogeneous linear equations in three variables x, y and z.

If we take k = 0 in (5), then $ax + by + cz = 0$ (7)

is called a homogeneous equation in three variables x, y and z.

If we take $k_1 = k_2 = k_3 = 0$ in (6) then

$$\left.\begin{aligned} a_1x+b_1y+c_1z=0 \\ a_2x+b_2y+c_2z=0 \\ a_3x+b_3y+c_3z=0 \end{aligned}\right\} \quad (8)$$

is called system of homogeneous linear equations in three variables x, y and z.

An order triple (t_1, t_2, t_3) is called a **solution** of system (6) if the equations are true for $x=t_1$, $y=t_2$ and $z=t_3$ The **solution set** is denoted by $S=\{(t_1, t_2, t_3)\}$. In the case of system (8), we see that it is always true for $x=t_1=0$, $y=t_2=0$ and $z=t_3=0$, so the order triple $(t_1, t_2, t_3)=(0,0,0)$ is a solution of the system. Such a solution is called the **trivial** (or **zero**) **solution** and any other solution, if it exists, other than trivial solution is called a **non-trivial** (or **non-zero**) solution of the system. Consider system (6). Since

$$\begin{bmatrix} a_1x+b_1y+c_1z \\ a_2x+b_2y+c_2z \\ a_3x+b_3y+c_3z \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

then system (6) may be written as a single matrix equation

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \quad (9)$$

or $\quad AX=B \quad (10)$

where, $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$

Did You Know ?

In writing the augmented matrix of a linear system, we enter zero whenever a variable is missing in equation, since the coefficient of the variable is zero.

A is called the **matrix of coefficients**, X is the column vector of variables and B is the column vector of constants. If we adjoin the column vector B of the constants to the matrix A on the right separated by a bar **or a vertical line**, that is

$$[A|B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \end{array}\right],$$

the new matrix so obtained is called **augmented matrix** of the given system.

2.6.2 Solution of three homogeneous linear equations in three unknowns

Consider the following system of three homogeneous linear equations in three unknowns x_1, x_2, x_3.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \quad & (i) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0 \quad & (ii) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0 \quad & (iii) \end{aligned}\right\} \quad (1)$$

which is equivalent to the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or simply AX = O,}$$

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $|A| \neq 0$, then A is non-singular and A^{-1} exists.

We have $A^{-1}(AX) = A^{-1}0 \Rightarrow (A^{-1}A)X = O \Rightarrow IX = O \Rightarrow X = O$, that is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $x_1 = 0, x_2 = 0$ and $x_3 = 0$. This shows that the system has only trivial solution. Thus, we may conclude "A system AX = O of three homogeneous linear equations in three variables has a trivial solution if A is non-singular i.e. $|A| \neq 0$".

Next we find the condition under which the system (1) has a non–trivial solution. Multiplying equations (i), (ii) and (iii) of the system by the cofactors A_{11}, A_{21} and A_{31} of the corresponding elements a_{11}, a_{21} and a_{31} and then adding them up, we get

$$(a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31})x_1 + (a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31})x_2 +$$
$$(a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31})x_3 = 0.$$

From this, we have $|A|x_1 = 0$. Likewise, we can have $|A|x_2 = 0$ and $|A|x_3 = 0$. The system (1) has a non-trivial solution if at least one of the variable x_1, x_2 and x_3 is different from zero. Suppose $x_1 \neq 0$, then $|A|\,x_1 = 0 \Rightarrow |A| = 0$. Thus, we may conclude: "A system AX = O of three homogeneous linear equations in three variables has a non-trivial solution if A is singular i.e. $|A| = 0$".

Example 11: Show that the following system has a trivial solution.

$$2x + y - z = 0 \qquad \text{(i)}$$
$$x + y - z = 0 \qquad \text{(ii)}$$
$$x + 2y + 2z = 0 \qquad \text{(iii)}$$

Solution: Since

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = 2+2 = 4 \neq 0,$$ the system has a trivial solution. Subtracting equation (ii) from (i), we get $x = 0$. Subtracting equation (iii) from (ii), we have $y = 3z$. Putting $x=0$ and $y=3z$ in equation (i) we obtain $z = 0$, and therefore from $y = 3z$, we get $y = 0$. Thus $x = 0, y = 0, z = 0$ and the system has only trivial solution.

Example 12: Show that the system ha non-trivial solution

$$x + y + 2z = 0 \qquad \text{(i)}$$
$$-2x + y - z = 0 \qquad \text{(ii)}$$
$$-x + 5y + 4z = 0 \qquad \text{(iii)}$$

Solution: Since

$$|A| = \begin{vmatrix} 1 & 1 & -2 \\ -2 & 1 & -1 \\ -1 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 3 & 3 \\ -1 & 6 & 6 \end{vmatrix} = 1\begin{vmatrix} 3 & 3 \\ 6 & 6 \end{vmatrix} = 18 - 18 = 0$$

Thus the given system has a non-trivial solution.

Adding 2 times equation (i) to (ii) we have $y = -z$

Subtracting equation (ii) from (i), we get $x = -z$ putting $x = -z = y$ in equation (iii) we have $-(-z) + 5(-z) + 4z = 0$ which is true for any value t of z. We get that $x = -t$, $y = -t$ and $z = t$ satisfy equations (i), (ii) and (iii) for any real value of t.

Thus the given system has infinitely many solutions.

Example 13: For what value of λ the system has a non-trivial solution. Solve the system for the value of λ.

$$x - y + 2z = 0$$
$$2x + y + \lambda z = 0$$
$$3x + y + 2z = 0$$

Solution: First we find the value of λ. We have $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & \lambda \\ 3 & 1 & 2 \end{bmatrix}$,

So $$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & \lambda \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & \lambda-4 \\ 3 & 4 & -4 \end{vmatrix} = 1.\begin{vmatrix} 3 & \lambda-4 \\ 4 & -4 \end{vmatrix} = -12-4(\lambda-4) = 4-4\lambda.$$

We know that the system has non-trivial solution if $|A|=0$, that is $4-4\lambda=0$ or $\lambda=1$. Substituting the value of λ into the system, we have

$$\begin{aligned} x-y+2z &= 0 \\ 2x+y+z &= 0 \\ 3x+y+2z &= 0 \end{aligned}$$

Now solving the first two equations, we get $x = -z$, $y = 3$. Putting these values in the third equation, we obtain $-3z+z+2z=0$ which is true for any value t of z. We see that $x=-t$, $y=t$ and $z=t$ satisfy all the three equations of the system for any real value of t. Thus the given system has infinitely many solutions for $\lambda=1$.

2.6.3 Consistency and inconsistency of a system

(a) A system of linear equations is said to be consistent if the system has only one (i.e. unique) solution or it has infinitely many solutions.

(b) A system of linear equations is said to be inconsistent if the system has no solution.

Consider the following three systems of linear equations in three variables.

$$\left.\begin{aligned} 2x+2y-z &= 4 \\ x-2y+z &= 2 \\ x+y &= 0 \end{aligned}\right\} \quad \text{(I)}$$

$$\left.\begin{aligned} x-2y+z &= 2 \\ -x-y+2z &= 1 \\ x-5y+4z &= 5 \end{aligned}\right\} \quad \text{(II)}$$

$$\left.\begin{aligned} x-2y+3z &= 1 \\ -2x+5y-4z &= -2 \\ x-4y-z &= 5 \end{aligned}\right\} \qquad \text{(III)}$$

We solve these systems now by performing the elementary row operations on the augmented matrices of these systems to reduce them to (row) echelon form.

(i) Consider system (I). the augmented matrix of the systems is

$$[A|B]=\left[\begin{array}{ccc|c} 2 & 2 & -1 & 4 \\ 1 & -2 & 1 & 2 \\ 1 & 1 & 0 & 0 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 2 & 2 & -1 & 4 \\ 1 & 1 & 0 & 0 \end{array}\right] \text{ by } R_1 \leftrightarrow R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 6 & -3 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \text{ by } R_2 - 2R_1$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 6 & -3 & 6 \\ 0 & 3 & -1 & -2 \end{array}\right] \text{ by } R_3 - R_1$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 6 & -3 & 0 \\ 0 & -6 & 2 & 4 \end{array}\right] \text{ by } -2R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 6 & -3 & 0 \\ 0 & 0 & -1 & 4 \end{array}\right] \text{ by } R_3 + R_2$$

The system (I) is reduced to **equivalent system,**

$x - 2y + z = 2$ (i)
$0 + 6y - 3z = 0$ (ii)
$0 + 0 - z = 4$ (iii)

Remember

Two systems of equations are said to be equivalent if they have the same solution set.

The system is now in triangular form. In this form the system can be easily solved. By equation (iii) we get $z = -4$.

Substituting the value of z in equation (ii) we get $y = -2$.

Now substituting the values of y and z in equation (i), we get $x = 2$. Thus the solution of the system is $x = 2$, $y = -2$ and $z = -4$. Since the system has a solution, so it is consistent.

(ii) Consider system (II). The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ -1 & -1 & 2 & 1 \\ 1 & -5 & 4 & 5 \end{array}\right]$$

then $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ -1 & -1 & 2 & 1 \\ 1 & -5 & 4 & 5 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -3 & 3 & 3 \\ 0 & -3 & 3 & 3 \end{array}\right]$ by $R_2 + R_1$ and $R_3 - R_1$

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ by $R_3 - R_2$

$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ by $\frac{1}{3}R_2$.

The system (II) is reduced to the equivalent system

$x - 2y + z = 2$ (i)

$-y + z = 1$ (ii)

$0z = 0$ (iii)

Equation (iii) is obviously satisfied for all choices of z. Equations (i) and (ii) yield

$x = 2 + 2y - z$ (iv)

$y = z - 1$ (v)

Since z is arbitrary, from equations (iv) and (v) we can find infinitely many values of x and y. This is equivalent to saying that the system has infinitely many solutions. Thus the system is consistent.

(3) Consider system (III). The augmented matrix of the system is

$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ -2 & 5 & -4 & -2 \\ 1 & -4 & -1 & 5 \end{array}\right]$ then $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ -2 & 5 & -4 & -2 \\ 1 & -4 & -1 & 5 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -4 & 4 \end{array}\right]$

by $R_2 + 2R_1$ and $R_3 - R_1$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right] \text{ by } R_3 + 2R_2$$

The system (III) is reduced to the equivalent system

$$x - 2y + 3z = 1 \quad \text{(i)}$$
$$y + 2z = 0 \quad \text{(ii)}$$
$$0z = 4 \quad \text{(iii)}$$

We see that the equation (iii) has no solution. Therefore, this system of equations has no solution. Hence the system is inconsistent.

From the above, we note that the system of linear equations may have no solution, have only one solution, or have infinitely many solutions.

2.6.4 Solution of a non-homogeneous linear equations

A system of non-homogeneous linear equations may be solved by using the following methods.

(a) Matrix Inversion Method i.e. $AX=B \Rightarrow X=A^{-1}B$

(b) Gauss Elimination Method (echelon form)

(c) Gauss–Jordan Method (reduced echelon form)

(d) Cramer's Rule.

(a) Matrix Inversion Method

Consider the following system of three non-homogeneous linear equations in three variables x_1, x_2 and x_3.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = k_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = k_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = k_3 \end{array}\right\}$$

This system is equivalent to the matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \textit{ or } AX = B, \textit{ where}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \textit{ and } B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}.$$

If A is non-singular, then A^{-1} exists. We have

$$AX = B \Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B.$$

Thus the matrix of variables is now determined as the product of $A^{-1}B$.

The method discussed above for finding the solution of a system of non–homogenous linear equations is known as matrix inversion method.

Example 14: Solve the system of equations by matrix **inversion** method

$$x_1 - 2x_2 + x_3 = 2$$
$$2x_1 + 2x_2 - x_3 = 4$$
$$x_1 + x_2 = 0$$

Solution: Since

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 0 \\ 2 & 2 & -1 \\ 1 & -2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 1 & -3 & 1 \end{vmatrix} = 1.\begin{vmatrix} 0 & -1 \\ -3 & 1 \end{vmatrix} = 3 \neq 0,$$

So, A^{-1} exists.

$$A^{-1} = \frac{1}{|A|} adj\, A = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 3 \\ 0 & -3 & 6 \end{bmatrix}$$

Did You Know

The matrix inversion method for solving a system of non-homogeneous linear equations is applicable only when the coefficient matrix A is non-singular i.e. $|A| \neq 0$.

$$But \quad X = A^{-1}B, so\ X = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 3 \\ 0 & -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1\times2+1\times4+0\times0 \\ -1\times2-1\times4+3\times0 \\ 0\times2-3\times4+6\times0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 \\ -6 \\ -12 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix},$$

that is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$. Thus $x_1 = 2, x_2 = -2$ *and* $x_3 = -4$

Which is the solution of the given system.

(b) Gauss elimination method (Echelon form)

We are already familiar with the method of reducing the augmented matrix of a system of non-homogeneous linear equations to echelon form. We now apply this method to find the solution of a system of non-homogeneous linear equations. The procedure is called Gauss Elimination Method (Echelon Form).

Example 15: Solve the following system by the method of echelon form.

$$\left.\begin{aligned} 2x_1 + 2x_2 - x_3 &= 4 \\ x_1 - 2x_2 + x_3 &= 2 \\ x_1 + x_2 &= 0 \end{aligned}\right\}$$

Solution: The augmented matrix of the given system is

$$\left[\begin{array}{ccc|c} 2 & 2 & -1 & 4 \\ 1 & -2 & 1 & 2 \\ 1 & 1 & 0 & 0 \end{array}\right]$$. By 2.6.3 (i) the echelon form of this matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 6 & -3 & 0 \\ 0 & 0 & -1 & 4 \end{array}\right].$$

From R_3, we have $x_3 = -4$.

From R_2, we have $6x_2 - 3x_3 = 0$

Substituting $x_3 = -4$, in this equation we get $x_2 = -2$.

From R_1, we have $x_1 - 2x_2 + x_3 = 2$

Now putting $x_2 = -2$ and $x_3 = -4$ we obtain $x_1 = 2$

Thus $x_1 = 2, x_2 = -2, x_3 = -4$ is the solution of the given system.

(c) Gauss–Jordan Method (Reduced Echelon Form)

Consider system of equations in example 14 above and the echelon form

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 6 & -3 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$ of its augmented matrix.

We reduce the matrix $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 6 & -3 & 0 \\ 0 & 0 & -1 & 4 \end{array}\right]$ to reduced (row) echelon form, that is

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 6 & -3 & 0 \\ 0 & 0 & -1 & 4 \end{array}\right] \underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 \end{array}\right] \text{by } \frac{1}{6}R_2 \text{ and } (-1)R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -4 \end{array}\right] by\ R_1 - R_3\ and\ R_2 + \frac{1}{2}R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -4 \end{array}\right] by\ R_1 + 2R_2$$

The equivalent system in the reduced (row) echelon form is

$x_1 = 2$, $x_2 = -2$, $x_3 = -4$.

which is the solution of the given system. The procedure illustrated above of transforming a system of non–homogeneous linear equations into an equivalent system in the reduced (row) **echelon form** is called the Gauss–Jordan Method (reduced echelon form).

(d) Cramer's Rule

Consider the following system of three non-homogeneous linear equations in three variables.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = k_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = k_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = k_3 \end{array}\right\} \quad (1)$$

which is equivalent to the matrix equation

$$AX = B \qquad (2)$$

Did You Know ?

Like matrix inversion method, the Cramer's rule is also applicable only when $|A| \neq 0$ Cramer's rule is simpler than matrix method for finding solution of the given system.

$$\text{where } A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B=\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}.$$

If $|A| \neq 0$, then A^{-1} exists and (2) can be written as $X = A^{-1}B$.

Since $A^{-1} = \frac{1}{|A|} adj\, A$, we have $X = A^{-1}B = \left(\frac{1}{|A|} adj\, A\right) B$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} A_{11}k_1 + A_{21}k_2 + A_{31}k_3 \\ A_{12}k_1 + A_{22}k_2 + A_{32}k_3 \\ A_{13}k_1 + A_{23}k_2 + A_{33}k_3 \end{bmatrix}$$

that is
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{A_{11}k_1 + A_{21}k_2 + A_{31}k_3}{|A|} \\ \frac{A_{12}k_1 + A_{22}k_2 + A_{32}k_3}{|A|} \\ \frac{A_{13}k_1 + A_{23}k_2 + A_{33}k_3}{|A|} \end{bmatrix}$$

Thus
$$x_1 = \frac{k_1A_{11} + k_2A_{21} + k_3A_{31}}{|A|} = \frac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}}{|A|},$$

$$x_2 = \frac{k_1A_{12} + k_2A_{22} + k_3A_{32}}{|A|} = \frac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{|A|},$$

$$x_3 = \frac{k_1A_{13} + k_2A_{23} + k_3A_{33}}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}}{|A|}.$$

This method of finding the solution of the system is called Cramer's Rule.

Example 16: Use Cramer's rule to solve the following system.

$$\left.\begin{array}{ll} x_1 - 2x_2 + x_3 & = 2 \\ 2x_1 + 2x_2 - x_3 & = 4 \\ x_1 + x_2 & = 0 \end{array}\right\}.$$

Solution: We have $|A| = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 3 \neq 0$

Now $x_1 = \dfrac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}}{|A|} = -\dfrac{\begin{vmatrix} 2 & -2 & 1 \\ 4 & 2 & -1 \\ 0 & 1 & 0 \end{vmatrix}}{3}$ (Expanding by third row)

$= -\dfrac{\begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix}}{3} \quad = \dfrac{6}{3} = 2,$

$x_2 = \dfrac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{|A|} = \dfrac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \\ 1 & 0 & 0 \end{vmatrix}}{3}$ (Expanding by third row)

$= \dfrac{\begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix}}{3} \quad = \dfrac{-6}{3} = -2,$

> **Note**
> We observe that the solution of the given system obtained by any of the above four methods are the same.

and $x_3 = \dfrac{\begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}}{|A|} = \dfrac{\begin{vmatrix} 1 & -2 & 2 \\ 2 & 2 & 4 \\ 1 & 1 & 0 \end{vmatrix}}{3} = -\dfrac{\begin{vmatrix} 1 & -3 & 2 \\ 2 & 0 & 4 \\ 1 & 0 & 0 \end{vmatrix}}{3}$ (Expanding by 2^{nd} Column)

$= 3\dfrac{\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix}}{3} \quad = -4$

Thus $x_1 = 2, x_2 = -2$ *and* $x_3 = -4$ is the solution of the given system.

EXERCISE 2.4

1. Solve the following system of equations by matrix inversion method.

(i)
$$4x-3y+z=11$$
$$2x+y-4z=-1$$
$$x+2y-2z=1$$

(ii)
$$x+y+z=1$$
$$x+y-2z=3$$
$$2x+y+z=2$$

2. Solve the following system of equations by the Gauss elimination method and Gauss–Jordan method.

(i)
$$x-y+4z=4$$
$$2x+2y-z=2$$
$$3x-2y+3z=-3$$

(ii)
$$2x+4y-z=0$$
$$x-2y-2z=2$$
$$-5x-8y+3z=-2$$

3. Use Cramer's rule to solve the following system of equations.

(i)
$$x-2y=-4$$
$$3x+y=-5$$
$$2x+z=-1$$

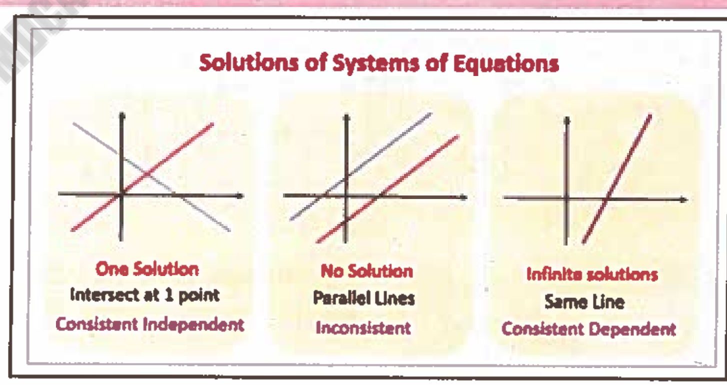
(ii)
$$x-y+2z=10$$
$$2x+y-2z=-4$$
$$3x+y+z=7$$

4. Solve the following system of homogeneous equations.

(i)
$$x_1-x_2+x_3=0$$
$$x_1+2x_2-x_3=0$$
$$2x_1+x_2+3x_3=0$$

(ii)
$$x_1+x_2+2x_3=0$$
$$-2x_1+x_2-x_3=0$$
$$-x_1+5x_2+4x_3=0$$

5. For what value of λ, the following system of homogeneous equations has a non-trivial solution. Solve the system.
$$x_1+5x_2+3x_3=0$$
$$5x_1+x_2-\lambda x_3=0$$
$$x_1+2x_2+\lambda x_3=0$$



REVIEW EXERCISE 2

1. Choose the correct options

(i) If $\begin{vmatrix} 7a-5b & 3c \\ -1 & 2 \end{vmatrix} = 0$, then which one of the following is correct?

(a) $14a + 3c = 5b$ (b) $14a - 3c = 5b$
(c) $14a+3c = 10b$ (d) $14a + 10b = 3c$

(ii) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and A_{ij} is the cofactor of a_{ij} in A. Then the value of $|A|$ is given by

(a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$ (b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$
(c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$ (d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

(iii) If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$ then the value of α is

(a) ± 1 (b) ± 2 (c) ± 3 (d) ±5

(iv) If $|A| = 47$, then find $|A^t|$

(a) −47 (b) 47 (c) 0 (d) Cannot be determined

(v) If det (A) = 5, then find det (15A) where A is of order 2x2.

(a) 225 (b) 75 (c) 375 (d) 1125

(vi) If $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then find A^n, (where $n \in N$)

(a) $\begin{bmatrix} 3n & 0 \\ 0 & 3n \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ (c) $3^n\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) I_{2x2}

2. Compute the product $\begin{bmatrix} -5 & 1 \\ 6 & -1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 4 & -5 & -1 \\ 5 & 6 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$.

3. Prove that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies $A^2 - 4A - 5I = 0$.

4. If $A=\begin{bmatrix}2 & 1 & 3\\ 0 & 4 & 6\\ 7 & 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}4 & 0 & -1\\ 2 & 0 & 3\\ -1 & 2 & 4\end{bmatrix}$. Find $|2A-B^2|$

5. Using properties of determinants, prove that

$$\begin{vmatrix}a^2+2a & 2a+1 & 1\\ 2a+1 & a+2 & 1\\ 3 & 3 & 1\end{vmatrix}=(a-1)^3$$

6. If $A=\begin{bmatrix}3 & 1 & -1\\ 0 & 1 & 2\end{bmatrix}$, then show that AA^t and A^tA are both symmetric.

7. *If* $A=\begin{bmatrix}4 & 3 & 3\\ -1 & 0 & -1\\ -4 & -4 & -3\end{bmatrix}$, prove that $A^{-1}=A$

8. If $A=\begin{bmatrix}4 & 3\\ -2 & 1\end{bmatrix}$, *then find* $A+10A^{-1}$

9. Solve the system
$$\begin{aligned}x+y+z&=4\\ 2x-3y+z&=2\\ -x+2y-z&=-1\end{aligned}$$

by using the following methods:

(i) Matrix Inversion (ii) Gauss Elimination

(iii) Gauss Jordan (iv) Cramer's Rule

Elementary row operations:

1. interchange of two rows

$$\begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 2 & 3\\ 5 & 5 & 1 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}5 & 5 & 1 & 0\\ 2 & 1 & 2 & 3\\ 1 & 2 & 3 & 4\end{bmatrix}$$

2. multiplication of a row by a non-zero number

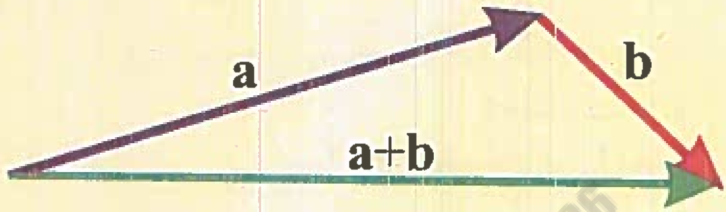
$$\begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 2 & 3\\ 5 & 5 & 1 & 0\end{bmatrix}\cdot 3 \Rightarrow \begin{bmatrix}1 & 2 & 3 & 4\\ 6 & 3 & 6 & 9\\ 5 & 5 & 1 & 0\end{bmatrix}$$

3. addition of a multiple of one row to another row

$$\begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 2 & 3\\ 5 & 5 & 1 & 0\end{bmatrix}\cdot 2 \Rightarrow \begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 2 & 3\\ 7 & 9 & 7 & 8\end{bmatrix}$$

UNIT 3

Vectors



STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Define a scalar and a vector.
- Give geometrical representation of a vector.
- Give the following fundamental definitions using geometrical representation.
  - magnitude of a vector,
  - equal vectors,
  - negative of a vector,
  - unit vector,
  - zero/null vector,
  - position vector,
  - parallel vectors,
  - addition and subtraction of vectors,
  - triangle, parallelogram and polygon laws of addition,
  - scalar multiplication.
- Represent a vector in a Cartesian plane by defining fundamental unit vectors i and j.
- Recognize all above definitions using analytical representation.
- Find a unit vector in the direction of another given vector.
- Find the position vector of a point which divides the line segment joining two points in a given ratio.
- Use vectors to prove simple theorems of descriptive geometry.
- Recognize rectangular coordinate system in space.
- Define unit vectors i, j and k.
- Recognize components of a vector.
- Give analytic representation of a vector.
- Find magnitude of a vector.
- Repeat all fundamental definitions for vectors in space which, in the plane, have already been discussed.
- State and prove
  - commutative law for vector addition.
  - associative law for vector addition.

STUDENTS LEARNING OUTCOMES

- Prove that:
  - 0 as the identity for vector addition.
  - –A as the inverse for A .
- State and prove:
  - commutative law for scalar multiplication,
  - associative law for scalar multiplication,
  - distributive laws for scalar multiplication.
- Define dot or scalar product of two vectors and give its geometrical interpretation.
- Prove that.
  - $i \cdot i = j \cdot j = k \cdot k = 1$,
  - $i \cdot j = j \cdot k = k \cdot i = 0$
- Express dot product in terms of components.
- Find the condition for orthogonality of two vectors.
- Prove the commutative and distributive laws for dot product.
- Explain direction cosines and direction ratios of a vector.
- Prove that the sum of the squares of direction cosines is unity.
- Use dot product to find the angle between two vectors.
- Find the projection of a vector along another vector.
- Find the work done by a constant force in moving an object along a given vector.
- Define cross or vector product of two vectors and give its geometrical interpretation.
- Prove that:
  - $i \times i = j \times j = k \times k = 0$,
  - $i \times j = -j \times i = k$,
  - $j \times k = -k \times j = i$,
  - $k \times i = -i \times k = j$.
- Express cross product in terms of components.
- Prove that the magnitude of A × B represents the area of a parallelogram with adjacent sides A and B .
- Find the condition for parallelism of two non-zero vectors.
- Prove that A × B = –B × A.
- Prove the distributive laws for cross product.
- Use cross product to find the angle between two vectors.
- Find the vector moment of a given force about a given point.
- Define scalar triple product of vectors.
- Express scalar triple product of vectors in terms of components (determinantal form).
- Prove that:
  - $i \cdot j \times k = j \cdot k \times i = k \cdot i \times j = 1$,
  - $i \cdot k \times j = j \cdot i \times k = k \cdot j \times i = -1$.
- Prove that dot and cross are inter-changeable in scalar triple product.
- Find the volume of
  - a parallelepiped,
  - a tetrahedron, determined by three given vectors.
- Define coplanar vectors and find the condition for coplanarity of three vectors.

3.1 Introduction

Physical quantities such as mass, temperature and work are measured by numbers referred to some chosen unit. These numbers are called **scalars**. Scalars being just numbers, can therefore be added, subtracted, multiplied and divided by using the fundamental laws of elementary algebra.

Other quantities exist such as displacement, velocity, acceleration and force, which require for their complete specification a direction as well as a scalar. These quantities are called **vectors** and may be represented by a straight line with an arrow. Vectors cannot be added, subtracted, multiplied or divided by ordinary mathematical rules but we use methods of vector addition (triangle rule or parallelogram rule) or other analytical methods for their multiplication, for this purpose.

Vectors have many applications in Geometry, Physics and Engineering. We begin with geometrical interpretation of a vector. However, in the sequel we shall apply vector methods to prove some fundamental results of descriptive geometry.

3.1.1 Scalar and Vector

Scalar Quantity: A quantity which has only magnitude and no direction is called a scalar quantity or simply a scalar.
Examples of scalar are mass, temperature, volume, work etc.

Vector Quantity: A quantity which has magnitude as well as direction is called a vector quantity or simply a vector.
Examples of vector are displacement, velocity, acceleration, force etc.

3.1.2 Geometrical representation of a vector

A vector is geometrically represented by an arrow or directed line segment say $\overrightarrow{OP}$, where the arrow indicates the direction of the vector and the length of the arrow specifies, on appropriate scale, the magnitude of the vector. The tail end O of the arrow is called its **origin** or **initial point** and the head (tip) P is called the **terminal point** or **terminus** (Figure 3.1)

In printed work, it is usual to denote all vectors by bold faced letters ***a***, ***b***, ***v*** etc. In hand written work, the vectors are denoted by $\vec{a}$, $\vec{b}$, $\vec{v}$ etc. The other notation used for vector is $\underline{a}, \underline{b}, \underline{v}$ etc.

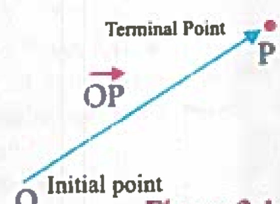


Figure 3.1

3.1.3 Fundamentals of a vector

(i) Magnitude of a vector

The magnitude or modulus of a vector $\overrightarrow{OA}$ or $\boldsymbol{a}$ is the length of the line segment representing the vector to the scale used. The magnitude of the vector $\overrightarrow{OA}$ is denoted by $|\overrightarrow{OA}|$, $|\boldsymbol{a}|$, $|\underline{a}|$ or a.

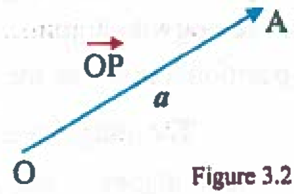


Figure 3.2

(ii) Equal vector

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be equal if they have the same magnitude and direction regardless of the position of their initial point. Symbolically, we write $\boldsymbol{a} = \boldsymbol{b}$ (Figure 3.3)

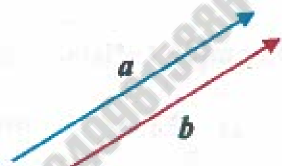


Figure 3.3

(iii) Negative of a vector

A vector having the same magnitude as another vector $\boldsymbol{a}$ but opposite in direction is called negative of a vector and is denoted by $-\boldsymbol{a}$ as shown in (Figure 3.4)

Figure 3.4

(iv) Zero vector or null vector

A vector which has zero magnitude and arbitrary direction is called the zero vector or null vector. Zero vector is denoted by O, $\vec{O}$ or $\underline{O}$.

(v) Unit vector

A vector whose magnitude is one is called unit vector. It is used to represent the direction of a vector. A unit vector is denoted by a letter with a hat over it, such as $\hat{a}$, $\hat{b}$, $\hat{v}$ etc. Any vector $\boldsymbol{a}$ can be written in terms of unit vector as $\boldsymbol{a} = |\boldsymbol{a}|\hat{a}$

Hence unit vector in the direction of $\boldsymbol{a}$ is obtained as $\hat{a} = \dfrac{a}{|a|}$

i.e. unit vector in a direction $= \dfrac{\text{Vector in that direction}}{\text{Modulus of the vector}}$

(vi) Parallel vectors

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{a} = \alpha \boldsymbol{b}$, where α is scalar. See for example (Figure 3.5)

Figure 3.5

For Your Information

The magnitude $|\boldsymbol{a}|$ of a vector $\boldsymbol{a}$ is a positive scalar quantity and therefore can be added, subtracted, multiplied and divided like all other scalar quantities.

(vii) Position Vector

A vector which joins a given point P in the plane or space with the origin is called position vector of the point P and is denoted by $\overrightarrow{OP}$ (Figure 3.6).

The magnitude of the position vector is equal to the distance between the given point and the origin and whose direction is the direction of the point from the origin.

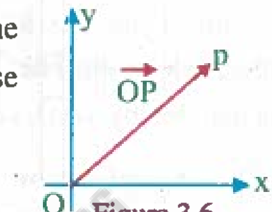


Figure 3.6

Example 1: Using graph paper, draw the vectors.

(a) $2\boldsymbol{a}$ (b) $-\boldsymbol{a}$ (c) $\frac{3}{4}\boldsymbol{a}$

where $\boldsymbol{a}$ is given in (Figure 3.7)

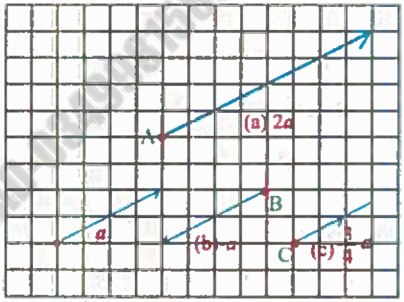


Figure 3.7

Solution: (a) The head of the vector $\boldsymbol{a}$ from its end point is 4 squares to the right and 2 squares up. Hence $2\boldsymbol{a}$ is 8 squares to the right and 4 squares up.

(b) $-\boldsymbol{a}$ is the negative of $\boldsymbol{a}$, so its direction is opposite to $\boldsymbol{a}$. Hence $-\boldsymbol{a}$ is 4 squares to the left and 2 squares down from its end point.

(c) $\frac{3}{4}\boldsymbol{a}$ is 3 squares to the right and 1 and a half squares up as shown in (Figure 3.7).

Example 2: In Figure 3.8, vectors $\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ are shown. State each of the vectors $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ and $\boldsymbol{s}$ in the form $k\boldsymbol{a}$.

Solution: The direction of $\boldsymbol{a}$ is 2 squares to the right and 4 squares up.

Hence $\quad \boldsymbol{p} = -\boldsymbol{a}\,,\ \boldsymbol{q} = \frac{1}{2}\boldsymbol{a}$

$\boldsymbol{r} = 2\boldsymbol{a}\,,\ \boldsymbol{s} = \frac{3}{2}\boldsymbol{a}$

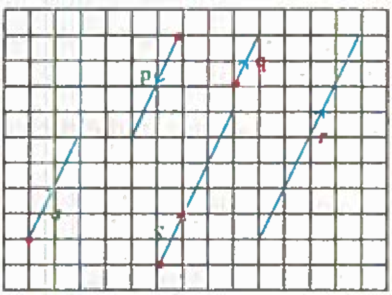


Figure 3.8

Example 3: What type of quadrilateral is ABCD, if (i) $\overrightarrow{AB} = \overrightarrow{CD}$ ii. $\overrightarrow{AB} = 3\overrightarrow{CD}$

Solution: (i) $\overrightarrow{AB} = \overrightarrow{CD}$ means that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal in length i.e. $|\overrightarrow{AB}| = |\overrightarrow{CD}|$ and $\overrightarrow{AB} \parallel \overrightarrow{CD}$. Hence ABCD is a parallelogram as shown in (Figure 3.9.)

Figure 3.9

(ii) $\overrightarrow{AB} = 3\overrightarrow{CD}$ means
$|\overrightarrow{AB}| = 3|\overrightarrow{CD}|$ and $\overrightarrow{AB} \parallel \overrightarrow{CD}$.
Hence ABCD is a trapezium as shown in (Figure 3.10.)

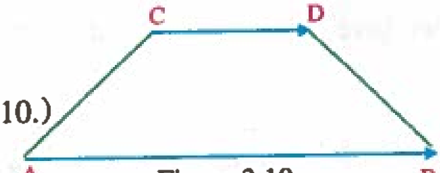


Figure 3.10

(viii) Addition and subtraction of vectors

(a) Addition of vectors

Any two vectors can be added by the following two laws.

- **Head – to – tail or Triangle law of addition**

To add two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ that is, to combine them into one vector, we draw them in such a way that the head of the first vector coincides with the tail of the second vector. The **sum** or **resultant vector $\boldsymbol{a+b}$** is obtained by joining the tail of the first vector with the head of the second vector as shown in (Figure 3.11).

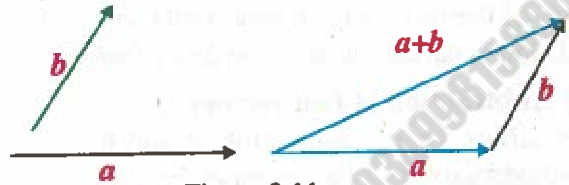


Figure 3.11

We call this way of adding the vectors as **Head–to–Tail or Triangle law of addition.**

- **Parallelogram Law of Addition**

If the two adjacent sides AB and AC of a parallelogram represent the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as shown in (Figure 3.12), then the diagonal AD represents the vector sum or resultant $\boldsymbol{a} + \boldsymbol{b}$ of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Thus $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC} = \boldsymbol{a} + \boldsymbol{b}$

We call this way of adding the vectors as the **parallelogram law of addition.**

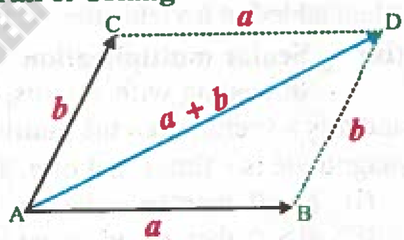


Figure 3.12

- **Polygon Law of Addition of Vectors**

The method of vector addition of two vectors can be extended to more than two vectors. Let **a, b, c, d** be four given vectors. Let O be any point and let us draw the vectors $\overrightarrow{OA}$ = a. From the terminal point A of the vector **a**, draw $\overrightarrow{AB}$ to represent vector **b**. From the terminal point B, draw $\overrightarrow{BC}$ to represent vector **c**. From the terminal point C, draw $\overrightarrow{CD}$ to represent vector **d**. Join OD. Then, from (Figure 3.13),

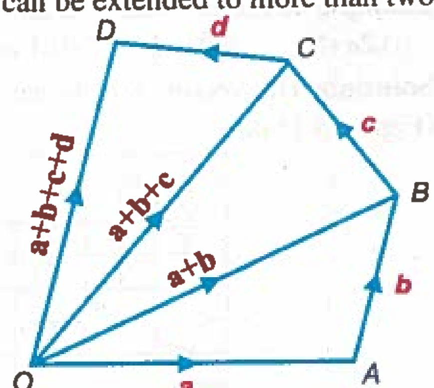


Figure 3.13

we have $\quad a+b+c+d = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$

$$= \overrightarrow{OB} + \overrightarrow{BC} + \overrightarrow{CD}$$

$$= \overrightarrow{OC} + \overrightarrow{CD} \qquad [\because \overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}]$$

$$= \overrightarrow{OD} \qquad [\because \overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OC}]$$

Thus the vector $\overrightarrow{OD}$ joining the initial point of the first vector **a** and the terminal point of the last vector **d** represents sum of the given vectors. This method of addition is called the polygon law of addition.

(b) Subtraction of two vectors

The difference of two vectors ***a*** and ***b***, denoted by ***a – b***, is the vector ***c*** obtained by adding vector ***a*** and the negative of ***b***, that is $\quad c = a - b = a + (-b)$

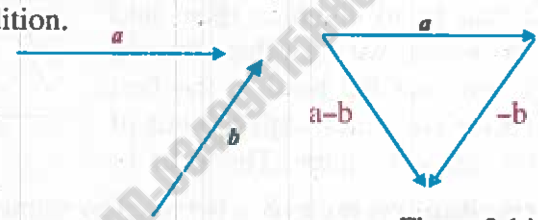


Figure 3.14

Thus, the difference ***a – b*** of vectors ***a*** and ***b*** is equal to a vector ***c*** which when added to ***b*** yields the vector ***a***. The difference ***a – b*** is shown in (Figure 3.14.)

(ix) Scalar multiplication

In dealing with vectors, we refer to real numbers as scalars. If k is a scalar and ***a*** is a vector, then the multiplication of ***a*** by k, denoted as ka, is a vector whose magnitude is k times that of ***a***. Thus, if

(i) $k = 0$, then ka is the zero vector

(ii) $k > 0$, then ***a*** and ka are in the same direction

(iii) $k < 0$, then ***a*** and ka are in the opposite direction

For illustration, see (Figure 3.15).

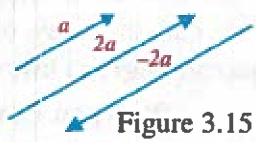


Figure 3.15

Example 4: For the vectors ***a*** and ***b*** given in (Figure 3.16 (a)) ,draw the vector

(i) ***2a+b*** (ii) ***a–b*** (iii) ***a–2b***

Solution: The vectors are shown in (Figure 3.16 (b))

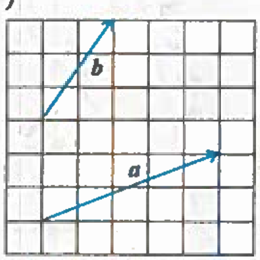


Figure 3.16 (a)

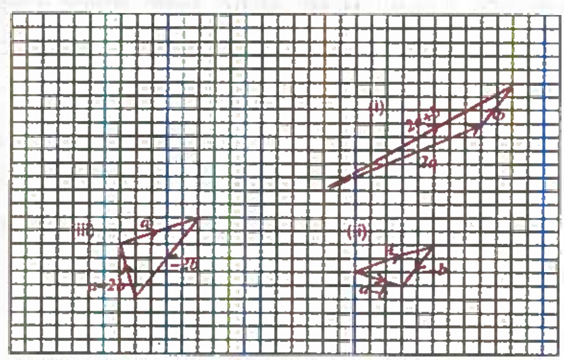


Figure 3.16 (b)

Draw vector $2a$ and from the head of $2a$ draw b. Then use head–to–tail rule to obtain $2a + b$.

(i) Draw a followed by $-b$, use triangle law of addition of vectors to obtain $a - b$.

(ii) Draw a followed by $-2b$, use triangle law of addition of vectors to obtain $a - 2b$.

Example 5: In ΔABC, $\overrightarrow{AB} = a$, $\overrightarrow{AC} = b$ and D is the midpoint of AB (Figure 3.17). State in terms of a, b. (i) $\overrightarrow{AD}$ (ii) $\overrightarrow{DC}$ (iii) $\overrightarrow{CD}$

Solution:

i) $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}a$

(ii) $\overrightarrow{DC} = \overrightarrow{AC} - \overrightarrow{AD} = b - \frac{1}{2}a$

(iii) $\overrightarrow{CD} = -\overrightarrow{DC} = \frac{1}{2}a - b$

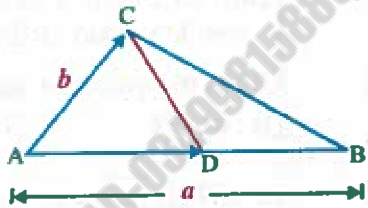


Figure 3.17

Theorem. For any vector a,

(i) The zero vector o has the property that $o + a = a + o = a$

(ii) The negative vector $-a$ of a has the property $a + (-a) = a - a = 0$

Proof. **(i) easy.**

If $\overrightarrow{OA} = \mathbf{a}$, we have, according to the definition of the multiplication of vectors by scalars, $\overrightarrow{AO} = (-1)\,\mathbf{a}$ Thus, $\mathbf{a} + (-1)\,\mathbf{a} = \overrightarrow{OA} + \overrightarrow{AO} = \overrightarrow{OO} = 0$

(ii) On account of this property, the vector $(-1)\,\mathbf{a}$ is called the negative of the vector $\mathbf{a}$, and we write $-\mathbf{a} = (-1)\,\mathbf{a}$

So that the relation $\mathbf{a} + (-1)\,\mathbf{a} = 0$,

may also be re-written as $\mathbf{a} + (-a) = 0$

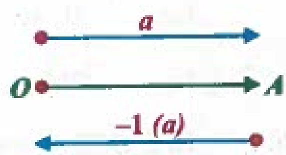
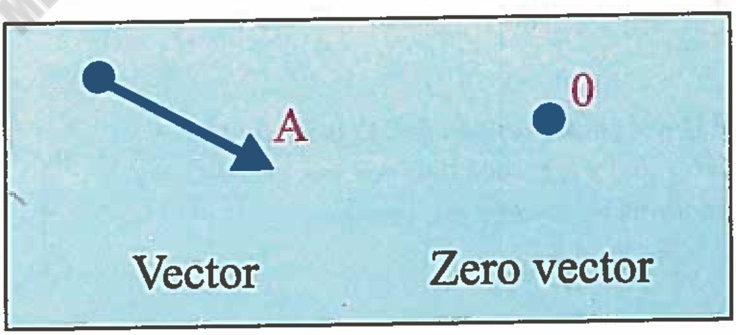
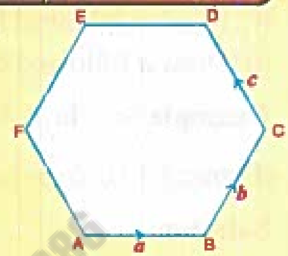


Figure 3.18



EXERCISE 3.1

1. ABCDEF is a regular hexagon $\overrightarrow{AB} = \boldsymbol{a}$, $\overrightarrow{BC} = \boldsymbol{b}$ and $\overrightarrow{CD} = \boldsymbol{c}$, state the following vectors as scalar multiple of $\boldsymbol{a}$, $\boldsymbol{b}$ or $\boldsymbol{c}$.



   (i) $\overrightarrow{DE}$ (ii) $\overrightarrow{EF}$ (iii) $\overrightarrow{FA}$ (iv) $\overrightarrow{AD}$ (v) $\overrightarrow{BE}$

   Hint: In a regular hexagon main diagonal $\overrightarrow{AD}$ is double the side $\overline{BC}$ and parallel to it.

2. Given the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as in Figure, draw the vectors:

   (i) $\boldsymbol{a} + 2\boldsymbol{b}$ (ii) $2\boldsymbol{a} - \boldsymbol{b}$ (iii) $3\boldsymbol{a} - 2\boldsymbol{b}$

3. In ΔOPQ, $\overrightarrow{OP} = \boldsymbol{p}$, $\overrightarrow{OQ} = \boldsymbol{q}$, R is the midpoint of $\overline{OP}$ and S lies on $\overline{OQ}$ such that $|\overline{OS}| = 3|\overline{SQ}|$. State in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$.

   (i) $\overrightarrow{OR}$ (ii) $\overrightarrow{PQ}$ (iii) $\overrightarrow{OS}$ (iv) $\overrightarrow{RS}$

4. OACB is a parallelogram with $\overrightarrow{OA} = \boldsymbol{a}$ *and* $\overrightarrow{OB} = \boldsymbol{b}$, $\overline{AC}$ is extended to D where $|AC| = 2|CD|$. Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

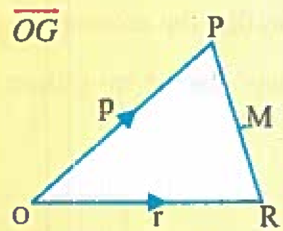
   (i) $\overrightarrow{AD}$ (ii) $\overrightarrow{OD}$ (iii) $\overrightarrow{BD}$

5. *OAB* is a triangle with $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$. M is the midpoint of OA and G lies on $\overline{MB}$ such that $|\overline{MG}| = \frac{1}{2}|\overline{GB}|$. State in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

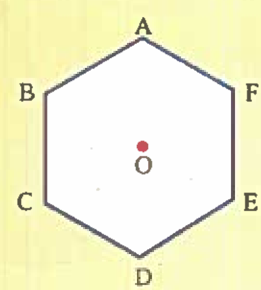
   (i) $\overrightarrow{OM}$ (ii) $\overrightarrow{MB}$ (iii) $\overrightarrow{MG}$ (iv) $\overrightarrow{OG}$

6. In Δ OPR, the mid-point of PR is M.
   If $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OR} = \mathbf{r}$, find in terms of p and r.

   (i) $\overrightarrow{PR}$ (ii) $\overrightarrow{PM}$ (iii) $\overrightarrow{OM}$



7. ABCDEF is a regular hexagon and O is its centre. The vectors x and y are such that $\overrightarrow{AB} = x$ and $\overrightarrow{BC} = y$. Express in terms of x and y the vectors $\overrightarrow{AC}$, $\overrightarrow{AO}$, $\overrightarrow{CD}$ and $\overrightarrow{BF}$.



3.1.4 Representation of a vector in a cartesian plane

We recall from our previous class that a **rectangular coordinate system** consists of two lines xx′ and yy′ drawn at right angle to each other as shown in (Figure 3.20), are known as **coordinate axes**. Their point of intersection is called **origin** and is denoted by O. The rectangular coordinate system is also called as **Cartesian coordinate system.**

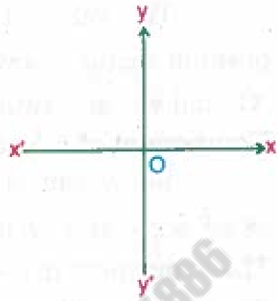


Figure 3.20

The horizontal line is called **x–axis** with positive direction to the right and the vertical line is called **y–axis** with positive direction upward. If P is a point in plane, it has two coordinates, one along x–axis and the other along y–axis. If the distances along x–axis and y–axis are determined by a and b respectively, then the point P is assigned an ordered pair of real numbers as (a,b) or P (a,b) as shown in (Figure 3.21). We call a and b the **x–coordinate** and **y–coordinate** of P.

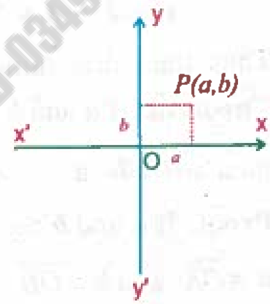
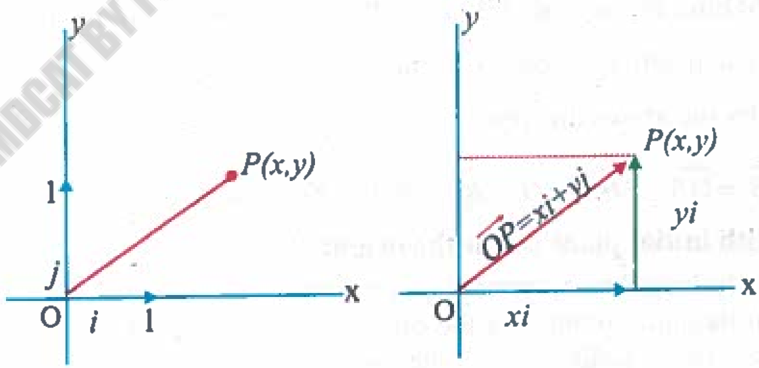


Figure 3.21

The set $\mathbb{R}^2 = \{(a,b): a,b \in \mathbb{R}\}$ is called the **Cartesian plane.** Thus an element $(a,b) \in \mathbb{R}^2$ represents a point $P(a,b)$ which is uniquely determined by its coordinates a and b.

In this section, we use rectangular coordinate system to represent a vector in the plane.

Let $\boldsymbol{i}$ denote the unit vector whose direction is along the positive x–axis and let $\boldsymbol{j}$ denote the unit vector whose direction is along the positive y–axis. Then every vector $\overrightarrow{OP}$ in the plane can be written uniquely in terms of the vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ as $\overrightarrow{OP} = \boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ where x and y are scalar. See (Figure 3.22).



(a) (b)

Figure 3.22

The vector $\overrightarrow{OP}$ is called the **position vector** of the point P. Thus, the position vector of any point $P(x,y)$ is the vector $\overrightarrow{OP}$ whose initial point is the origin 'O' and whose terminal point is P.

Component of a Vector

In the representation of the position vector to any point $P(x,y)$ in the plane as $\overrightarrow{OP} = \boldsymbol{r} = xi + yj$, the scalars x and y are called the **components** of the vector $\boldsymbol{r}$. The component in i–direction is x, while the component in j–direction is y.

For example, if P(5,–4) be a point in the plane. Then the vector $\boldsymbol{r}$ represented by the position vector to the point P(5,–4) is

$$\boldsymbol{r} = xi + yj = 5i + (-4)j.$$

Thus, the i–direction component is 5 and the j–direction component is –4.

Theorem: If $\boldsymbol{a}$ and $\boldsymbol{b}$ are position vectors of points A and B respectively,

then $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

Proof: If $\boldsymbol{a}$ and $\boldsymbol{b}$ are position vectors of the points A and B respectively, then

$\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$ (Figure 3.23)

Using triangle law of vector additions, we have

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\Rightarrow \overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$$

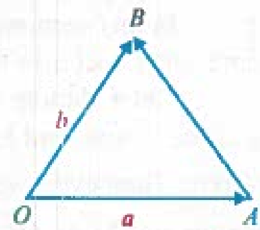


Figure 3.23

Example 6: Find the vector $\overrightarrow{AB}$ from the point A (–4,6) to the point B (6,8).

Solution: The position vectors of A and B are $\overrightarrow{OA} = -4i + 6j$ and $\overrightarrow{OB} = 6i + 8j$.

Therefore by the above theorem

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (6i + 8j) - (-4i + 6j) = 6i + 8j + 4i - 6j = 10i + 2j$$

Vectors with initial point not at the origin

We defined the component of a vector to be the coordinates of its terminal point when its initial point is at the origin. Now we will find the components of a vector whose initial point is not at the origin.

Suppose $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are two points in the plane. Suppose $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ be the position vectors of P_1 and P_2 as shown in (Figure 3.24).

Then $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$

$= (x_2 i + y_2 j) - (x_1 i + y_1 j)$

$= (x_2 - x_1)i + (y_2 - y_1)j$

Thus the i–component is $x_2 - x_1$ and the j–component is $y_2 - y_1$

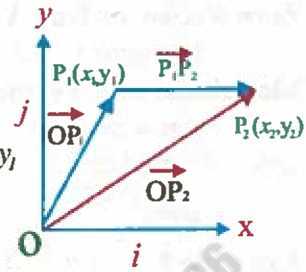


Figure 3.24

3.1.5 Algebra of Vectors

In this section we define addition, subtraction, scalar multiplication, and so on, for vectors in plane.

Equal Vectors

Two vectors $u = x_1 i + y_1 j$ and $v = x_2 i + y_2 j$ are said to be equal if and only if they have the same components that is

$u = v$ if and only if $x_1 = x_2$ and $y_1 = y_2$

Example 7: If $u = 2i + yj$ and $v = xi - j$, then find x and y.

Solution: $u = v$ or $2i + yj = xi - j$

By comparison we have $x = 2$ and $y = -1$

Addition of Vectors

If $u = x_1 i + y_1 j$ and $v = x_2 i + y_2 j$ are two vectors, then their addition, denoted by $u + v$, is defined as $u + v = (x_1 + x_2)\, i + (y_1 + y_2) j$

Thus, to add two vectors, we add their corresponding components.

Scalar Multiplication

The multiplication of the vector $u = xi + yj$ by a scalar k, that is ku is defined as

$ku = k(xi + yj) = (kx)i + (ky)j$

Negative of a Vector

If $u = xi + yj$ is a vector, then negative of u, denoted by $-u$, is defined as

$-u = -(xi + yj) = -xi - yj$

Thus, if we take $k = -1$ in the definition of scalar multiplication, we obtain $-u$ that is the negative of the vector u.

Subtraction of Vector

If $u = x_1 i + y_1 j$ and $v = x_2 i + y_2 j$ are two vectors, then their difference, denoted by $u - v$, is defined as $u - v = (x_1 - x_2)\, i + (y_1 - y_2) j$

Thus, to subtract two vectors, we subtract their corresponding components.

Example 8: If $u = 3i + 4j$ and $v = 4i - 5j$,

Find (i) $u + v$ (ii) $2u$ (iii) $-v$ (iv) $2u - 3v$

Solution:

(i) $u + v = (3i + 4j) + (4i - 5j) = (3 + 4)i + [4 + (-5)]\, j = 7i - j$

(ii) $2u = 2(3i + 4j) = (2\cdot 3)i + (2\cdot 4)j = 6i + 8j$

(iii) $-v = -(4i - 5j) = -4i - (-5)j = -4i + 5j$

(iv) $2u - 3v = 2(3i + 4j) - 3(4i - 5j) = 6i + 8j - 12i + 15j = -6i + 23j$

Zero Vector or Null Vector

The zero vector or null vector is denoted by O and is defined as $0 = 0i + 0j$

Magnitude of a Vector

If $u = xi + yj$ is a vector, its **magnitude** or **norm** or **length** is denoted by $|u|$ and is defined as

$$|u| = \sqrt{x^2 + y^2}$$

Example 9: If $u = 2i - 3j$, then find $|u|$.

Solution: $|u| = \sqrt{(2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$

Unit Vector

If the magnitude of the given vector $u = xi + yj$ is 1, it is called a unit vector. That is, u is a unit vector if $|u| = 1$

Properties of Magnitude of a Vector

Theorem If $u = xi + yj$ is a vector and k is a scalar, then

(i) $|u| \geq 0$ (ii) $|u| = 0$ if and only if $u = 0$ (zero vector)

(iii) $|-u| = |u|$ (iv) $|ku| = |k|\,|u|$

Proof.

(i) $|u| = \sqrt{x^2 + y^2} \geq 0$ for all x and y.

(ii) $|u| = \sqrt{x^2 + y^2} = 0$ if and only if $x = 0$ and $y = 0$

if and only if $u = 0i + 0j$

if and only if $u = o$ (zero vector)

(iii) $|-u| = |-xi - yj| = \sqrt{(-x)^2 + (-y)^2} = \sqrt{x^2 + y^2} = |u|$

(iv) $|ku| = |(kx)i + (ky)j| = \sqrt{(kx)^2 + (ky)^2} = \sqrt{k^2(x^2 + y^2)}$

$= \sqrt{k^2}\,\sqrt{x^2 + y^2} = |k|\,|u|$

3.1.6 A Unit Vector in the direction of another Vector

If $u = xi + yj$ is a vector with magnitude $|u| \neq 0$, then $\frac{u}{|u|}$ is a unit vector whose direction is the same as that of u. It is usual to denote a unit vector in the direction of vector u by $\hat{u}$.

Clearly any vector u can be written in terms of unit vector as $u = |u|\,\hat{u}$

Hence a unit vector in the direction of u is given by

$$\hat{u} = \frac{u}{|u|} \Rightarrow \quad \hat{u} = \frac{xi + yj}{\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}}i + \frac{y}{\sqrt{x^2 + y^2}}j.$$

Example 10: Find a unit vector in the same direction as the vector $3i-2j$

Solution: Let $u = 3i - 2j$

Then $|u| = \sqrt{(3)^2+(-2)^2} = \sqrt{9+4} = \sqrt{13}$

Since $\hat{u} = \frac{u}{|u|}$, so $\hat{u} = \frac{3i-2j}{\sqrt{13}} = \frac{3}{\sqrt{13}}i - \frac{2}{\sqrt{13}}j.$

Note

The vector $\hat{u}$ is in fact a unit vector, because by property (iv) of magnitude of a vector

$|\hat{u}| = \left|\frac{u}{|u|}\right| = \frac{|u|}{|u|} = 1$

Notation for Vectors in Coordinate System

Sometimes we use the notation $[x,y]$ or $\langle x,y\rangle$ for the vector $r = xi + yj$ which has its initial point at the origin of the rectangular coordinate system. The terminal point of r will have coordinates of the form (x,y). We call these coordinates the components of r. In this notation, the unit vectors i and j are given by $i = [1,0]$, $j = [0,1]$. If $r_1 = [x_1, y_1]$ and $r_2 = [x_2, y_2]$ are vectors and k any scalar, then addition and scalar multiplication are defined as $r_1 + r_2 = [x_1,y_1] + [x_2,y_2] = [x_1 + x_2, y_1 + y_2]$ and $kr_1 = k[x_1,y_1] = [kx_1, ky_1]$
Using the definition of addition and scalar multiplication, the vector $r = xi + yj$ can be written as

$$r = xi + yj = x[1,0] + y[0,1] = [x,0] + [0,y] = [x,y]$$

Thus $r = xi + yj = [x,y]$

3.1.7 Ratio Formula

Theorem: Let a and b be the position vectors of the points A and B respectively. If C divides AB internally in the ratio $p:q$, then the position vector c of C is given

by $c = \frac{qa + pb}{q+p}$

Proof: If C divides the line segment $\overrightarrow{AB}$ internally in the ratio $p:q$, then $\frac{\overline{AC}}{\overline{CB}} = \frac{p}{q}$ as shown in the (Figure 3.25).

Hence $q\overrightarrow{AC} = p\overrightarrow{CB} \Rightarrow q(c-a) = p(b-c)$

$\Rightarrow qc - qa = pb - pc \Rightarrow qc + pc = qa + pb$

$\Rightarrow (q+p)c = qa + pb \Rightarrow c = \frac{qa + pb}{q+p}$

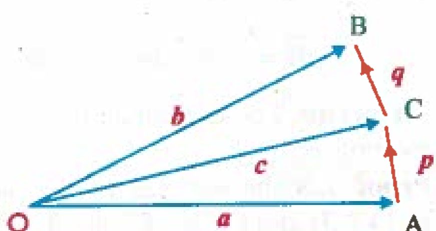


Figure 3.25

Corollary: If p : q = 1 : 1, then C is the midpoint of AB and its position vector c is given by $c = \frac{a+b}{2}$

Example 11: Find the position vector of the point dividing the join of point A with position vector $2i-3j$ and point B with position vector $3i+2j$ in the ratio 4:3

Solution: Suppose that the position vectors of the points A and B are $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. Then $\boldsymbol{a} = 2i - 3j$ and $\boldsymbol{b} = 3i + 2j$

Suppose that $\boldsymbol{c}$ is the position vector of the point C that divides the segment AB in ratio 4:3.

Then by ratio theorem (theorem above)

$$c = \frac{3a+4b}{3+4} = \frac{3(2i-3j)+4(3i+2j)}{7} = \frac{6i-9j+12i+8j}{7} = \frac{18i-j}{7} = \frac{18}{7}i - \frac{1}{7}j$$

3.1.8 Application to Geometry

In this section, we shall use vectors to prove some basic theorems of geometry.

Theorem: Prove that the straight line joining the midpoints of the two sides of a triangle is parallel to the third side and equal to one half of it.

Proof: Let OAB be a triangle and D,E be the midpoints of sides OA and OB respectively (see **Figure 3.26**)

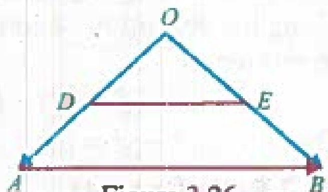


Figure 3.26

Let $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$, then

$\overrightarrow{OD} = \frac{a}{2}$, $\overrightarrow{OE} = \frac{b}{2}$ $\because$ D & E are the mid-points of $\overrightarrow{OA}$ & $\overrightarrow{OB}$ respectively

Now $\overrightarrow{DE} = \overrightarrow{DO} + \overrightarrow{OE} = -\overrightarrow{OD} + \overrightarrow{OE}$

$$= \frac{-a}{2} + \frac{b}{2} = \frac{b-a}{2} \qquad (1)$$

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = -a + b = b - a \qquad (2)$$

Therefore from (1) and (2), we have

$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AB}$ Hence $\overrightarrow{DE} \parallel \overrightarrow{AB}$ and $\overrightarrow{DE}$ is equal to one half of $\overrightarrow{AB}$

Theorem: The diagonals of a parallelogram bisect each other.

Proof: Let the vertices of the parallelogram be O,A,B and C (See **Figure 3.27**)

Let $\boldsymbol{a}$, $\boldsymbol{b}$ be the position vectors of A and B respectively.

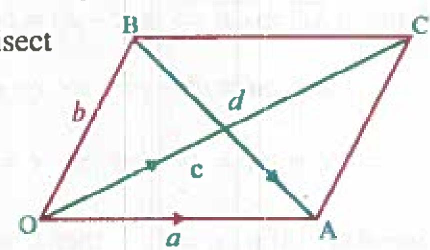


Figure 3.27

Then $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$.

By addition of vectors, we have $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{b}$

The midpoint of the diagonal $\overrightarrow{OC}$ has the position vector

$$c = \frac{\overrightarrow{OC}}{2} = \frac{a+b}{2} \qquad (1)$$

Again by addition of vectors, we have $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = b - a$

The midpoint of the diagonal $\overrightarrow{AB}$ has the position vector

$$d = \overrightarrow{OA} + \frac{\overrightarrow{AB}}{2} = a + \frac{b-a}{2} = \frac{2a+b-a}{2} = \frac{a+b}{2} \qquad (2)$$

From (1) and (2), we have $c=d$.

This shows that the midpoints of the diagonal $\overrightarrow{OC}$ and $\overrightarrow{AB}$ are the same.

Thus the diagonals of the parallelogram bisect each other.

3.2 Vectors in Space

In section 3.1.4 we discussed vectors in the plane. In this section, we again consider vectors, but vectors in space.

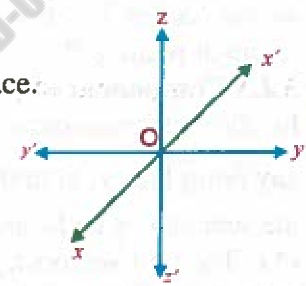


Figure 3.28

3.2.1 Rectangular coordinate system in space

In space, a rectangular coordinate system (or Cartesian coordinate system) consists of three mutually perpendicular lines through a common point O. The point O is called **origin** and the mutually perpendicular coordinate lines xox', yoy' and zoz' are respectively x–, y– and z–axis (Figure 3.28). The positive x–axis points towards the reader, the y–axis to the right and z–axis points upwards.

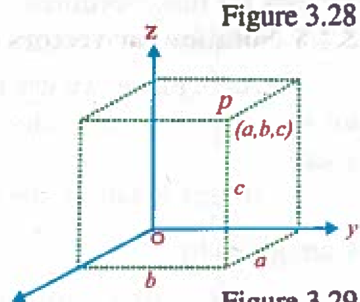


Figure 3.29

The coordinate axes, taken in pair, determine **three coordinate planes** namely the **xy–plane**, the **xz–plane** and the **yz–plane**. If the distances along x–, y– and z– axes are denoted by *a, b, c*, then the point P is assigned an ordered triple of real numbers as *(a, b, c)* or P*(a, b, c)* as shown in Figure 3.29. We call *a*, *b* and *c* the **x–coordinate, y–coordinate** and **z–coordinate** of P. Hence the point P whose coordinates are (4, 5, 6) is 4 units from O in the direction of $\overrightarrow{ox}$. 5 units from O in the direction of $\overrightarrow{oy}$. 6 units from O in the direction of oz as shown in (Figure 3.30).

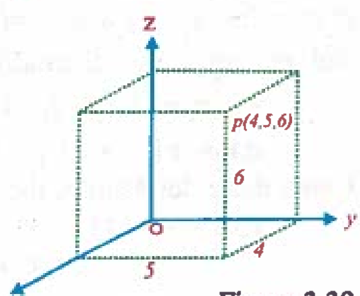


Figure 3.30

The set $\mathbb{R}^3 = \{(a, b, c): a, b, c \in \mathbb{R}\}$ is called the **three– dimensional space (or 3–dimensional space).**

3.2.2 Vectors in three dimensional space

Let ***i***, ***j*** and ***k*** be three mutually perpendicular unit vectors in the direction of coordinate axes as follows:

i is a unit vector along positive x–axis, ox

j is a unit vector along positive y–axis, oy

k is a unit vector along positive z–axis, oz

as shown in **Figure 3.31.**

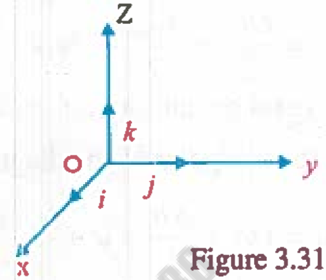


Figure 3.31

If P(*x, y, z*) is any point in the space, then the **position vector** $\overrightarrow{OP}$ of the point P can be written in the form.

$\overrightarrow{OP} = \boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ as shown in **Figure 3.32.**

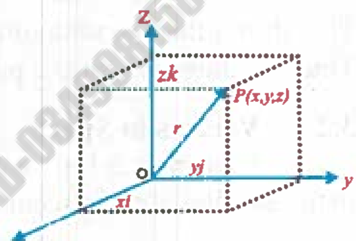


Figure 3.32

Thus, a position vector of the point P is a vector $\overrightarrow{OP}$ whose initial point is at the origin O and whose terminal point is P.

3.2.3 Component of a Vector

In the representation of the position vector to any point P(x, y, z) in the space as $\overrightarrow{OP} = \boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, the scalars x, y and z are called the **components** of ***r***. The unit vectors ***i***, ***j*** and ***k*** are the **unit base vectors** for this coordinate system.

Did You Know

If $\overrightarrow{P_1P_2}$ is a vector in space with initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$, Then $\overrightarrow{P_1P_2} = (x_2-x_1)\boldsymbol{i} + (y_2-y_1)\boldsymbol{j} + (z_2-z_1)\boldsymbol{k}$. So the components of $\overrightarrow{P_1P_2}$ in ***i***, ***j*** and ***k*** directions are $(x_2-x_1), (y_2-y_1), (z_2-z_1)$ respectively.

3.2.4 Notation for vectors in coordinate system

As in plane, we use the notation *[x, y, z]* or *<x, y, z>* for the vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ in space.

In this notation, the unit vectors ***i***, ***j*** and ***k*** are given by

$$\boldsymbol{i} = [1,0,0], \boldsymbol{j} = [0,1,0], \boldsymbol{k} = [0,0,1]$$

If $\boldsymbol{r_1} = [x_1, y_1, z_1]$ and $\boldsymbol{r_2} = [x_2, y_2, z_2]$ are vectors and α any scalar, then addition and scalar multiplication is defined as

$$\boldsymbol{r_1} + \boldsymbol{r_2} = [x_1, y_1, z_1] + [x_2, y_2, z_2] = [x_1+x_2, y_1+y_2, z_1+z_2] \text{ and}$$

$$\alpha\, \boldsymbol{r_1} = \alpha\, [x_1, y_1, z_1] = [\alpha\, x_1, \alpha\, y_1, \alpha\, z_1]$$

Using these definitions, the vector $\boldsymbol{r} = x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ can be written as

$$\boldsymbol{r} = x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k} = x[1,0,0] + y[0,1,0] + z[0,0,1]$$

$$= [x,0,0] + [0,y,0] + [0,0,z] = [x,y,z]$$

Thus $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = [x,y,z]$

3.2.5 Magnitude of a Vector

The magnitude or norm or length $|u|$ of a vector $u = xi + yj + zk$ in the space is the distance of the point $P(x, y, z)$ from the origin. That is $|u| = \sqrt{x^2 + y^2 + z^2}$

Unit Vector

If the magnitude of the vector $u = xi + yj + zk$ is 1, it is called a unit vector. That is $|u| = 1$

Example 12: If $u = 2i - j + 3k$, $v = i + j - k$, then find

(i) $u+2v$ (ii) $3u-2v$ (iii) $3(u-2v)$

(iv) $|u+v|$ (v) $|u|+|v|$ (vi) $\dfrac{u}{|u|}$

Solution:

(i) $u+2v = (2i-j+3k)+2(i+j-k) = 2i-j+3k+2i+2j-2k = 4i+j-k$

(ii) $3u-2v = 3(2i-j+3k)-2(i+j-k) = 6i-3j+9k-2i-2j+2k = 4i-5j+11k$

(iii) $3(u-2v) = 3[(2i-j+3k)-2(i+j-k)] = 3[2i-j+3k-2i-2j+2k]$

$= 3(0i-3j+5k) = -9j+15k$

(iv) $|u+v| = |(2i-j+3k)+(i+j-k)| = |2i-j+3k+i+j-k| = |3i+0j+2k|$

$= \sqrt{(3)^2+(0)^2+(2)^2} = \sqrt{9+4} = \sqrt{13}$

(v) $|u|+|v| = \sqrt{(2)^2+(-1)^2+(3)^2} + \sqrt{(1)^2+(1)^2+(-1)^2}$

$= \sqrt{4+1+9} + \sqrt{1+1+1} = \sqrt{14} + \sqrt{3}$

(vi) $\dfrac{u}{|u|} = \dfrac{2i-j+3k}{\sqrt{13}} = \dfrac{2}{\sqrt{13}}i - \dfrac{1}{\sqrt{13}}j + \dfrac{3}{\sqrt{13}}k$

3.2.6 Algebra of Vectors

In this section we define addition, subtraction scalar multiplication etc of vectors. Our definitions are the same as given for plane vectors except that in this case we consider vectors in space.

Equal Vectors

Two vectors $u = x_1i + y_1j + z_1k$ and $v = x_2i + y_2j + z_2k$ are said to be equal if and only if they have the same components.

That is $u = v$ if and only if $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$

Addition of Vectors

The addition of two vectors $u = x_1i + y_1j + z_1k$ and $v = x_2i + y_2j + z_2k$ is defined as

$u + v = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$

That is, to add two vectors, we add their corresponding components.

Scalar Multiplication

The scalar multiplication $\alpha\, \boldsymbol{u}$ of a vector $\boldsymbol{u}=xi+yj+zk$ by a scalar α is defined as

$\alpha\, \boldsymbol{u} = \alpha\,(xi + yj + zk) = (\alpha\, x)\,i + (\alpha\, y)j + (\alpha\, z)k$

Negative of a Vector

The negative of a vector $\boldsymbol{u} = xi + yj + zk$ is defined as

$$-\boldsymbol{u} = -(xi + yj + zk) = -xi - yj - zk$$

Subtraction of a Vector

The difference of two vectors $\boldsymbol{u} = x_1 i + y_1 j + z_1 k$ and $\boldsymbol{v} = x_2 i + y_2 j + z_2 k$ is defined as $\quad \boldsymbol{u} - \boldsymbol{v} = (x_1 - x_2)i + (y_1 - y_2)j + (z_1 - z_2)k$

That is, to subtract two vectors, we subtract their corresponding components.

Zero Vector or Null Vector

The zero vector or null vector O is defined as $\quad 0 = 0i + 0j + 0k$

Properties of Vectors

The following properties hold for vectors in plane as well as in space.

Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be vectors and let α and β be scalars, then

(i)	$u+v=v+u$	(commutative property for addition)
(ii)	$(u+v)+w=u+(v+w)$	(Associative property for addition)
(iii)	$u+o=o+u=o$	(Identity for vector addition)
(iv)	$u+(-u)=o$	(Inverse for vector addition)
(v)	$\alpha\,(\beta u) = (\alpha\beta)u$	(Associative property for scalar multiplication)
(vi)	$\alpha\,(u+v) = \alpha u + \alpha v$	(Distributive property of scalar multiplication over vector addition)
(vii)	$(\alpha+\beta)\, u = \alpha u + \beta u$	(Distributive property of vector multiplication over scalar addition)
(viii)	$1u = u$	

Application to Geometry

Distance between two points in Space

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space. Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be the position vectors of A and B (Figure 3.33). Then

$\overrightarrow{OA} = x_1 i + y_1 j + z_1 k,\ \overrightarrow{OB} = x_2 i + y_2 j + z_2 k$

Now $\quad \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (x_2 - x_1)\, i + (y_2 - y_1)j + (z_2 - z_1)k$

Hence $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

which is called the **distance formula.**

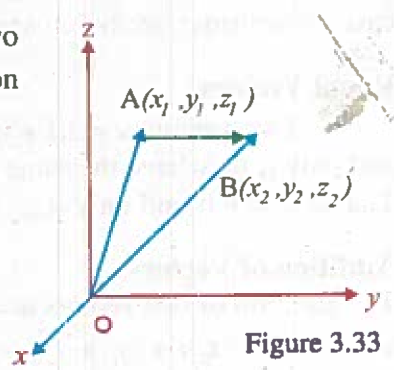


Figure 3.33

Theorem: Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space. The coordinate of point C which divides $\overrightarrow{AB}$ in the ratio $m_1:m_2$ are

$$\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}, \frac{m_1z_2+m_2z_1}{m_1+m_2}\right)$$

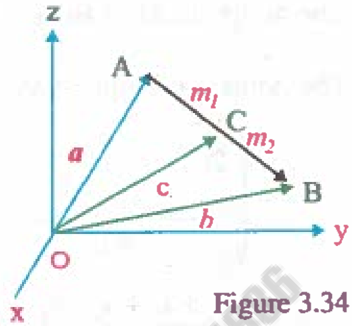


Figure 3.34

Proof. Let $C(x, y, z)$ divides $\overrightarrow{AB}$ in the ratio $m_1:m_2$ internally (Figure 3.34). If $\boldsymbol{a}$ is the position vector of A and $\boldsymbol{b}$ is the position vector of B, then the position vector $\boldsymbol{c}$ of C already found in Ratio theorem is

$$c = \frac{m_1b+m_2a}{m_1+m_2}$$

$$\therefore \quad xi + yj + zk = \frac{1}{m_1+m_2}\,[m_1(x_2i+y_2j+z_2k)+m_2(x_1i+y_1j+z_1k)]$$

$$= \frac{(m_1x_2+m_2x_1)i+(m_1y_2+m_2y_1)j+(m_1z_2+m_2z_1)k}{m_1+m_2}$$

$$= \frac{m_1x_2+m_2x_1}{m_1+m_2}i+\frac{m_1y_2+m_2y_1}{m_1+m_2}j+\frac{m_1z_2+m_2z_1}{m_1+m_2}k$$

Comparing the corresponding components on both sides,

$$\Rightarrow \quad x = \frac{m_1x_2+m_2x_1}{m_1+m_2},\ y = \frac{m_1y_2+m_2y_1}{m_1+m_2},\ z = \frac{m_1z_2+m_2z_1}{m_1+m_2}$$

Thus $C(x,y,z) = C\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}, \frac{m_1z_2+m_2z_1}{m_1+m_2}\right)$

Did You Know

If λ is negative, the point C divides $\overline{AB}$ externally in the ratio $\lambda:1$

Corollary: If $\frac{m_1}{m_2} = \lambda$, then the point C divides $\overrightarrow{AB}$ in the ratio $\lambda:1$ and

$$x = \frac{\lambda_1x_2+x_1}{1+\lambda},\ y = \frac{\lambda_1y_2+y_1}{1+\lambda},\ z = \frac{\lambda_1z_2+z_1}{1+\lambda}$$

Theorem: Prove that the coordinates of the centroid of a triangle ABC with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right).$$

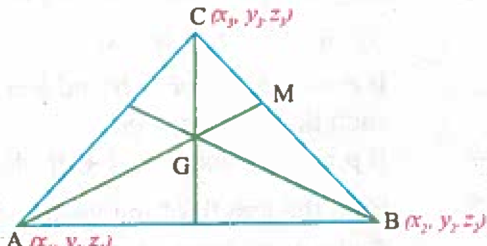


Figure 3.35

Proof: The centroid of the triangle ABC is the point G where all the three medians intersect each other in the ratio 2:1 (see Figure 3.35)

The midpoint M of BC has the coordinates $M\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$.

The point G dividing $\overrightarrow{AM}$ in the ratio AG:GM = 2:1 has the coordinates

$$\left(\frac{\frac{2(x_2+x_3)}{2}+1\cdot x_1}{2+1}, \frac{\frac{2(y_2+y_3)}{2}+1\cdot y_1}{2+1}, \frac{\frac{2(z_2+z_3)}{2}+1\cdot z_1}{2+1}\right)$$

$$=\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Example 13: Find the length of the median through O of the triangle OAB, where A is the point (2, 7, −1) and B is the point (4, 1, 2)

Solution: Let OAB be a triangle as shown in (Figure 3.36).

The coordinates of M the midpoint of AB are

$$\left(\frac{2+4}{2}, \frac{7+1}{2}, \frac{-1+2}{2}\right) = (3, 4, \frac{1}{2})$$

So the length of $\overline{OM}$ is

$$|\overline{OM}| = \sqrt{(3)^2+(4)^2+(\frac{1}{2})^2} = \frac{\sqrt{101}}{2}$$

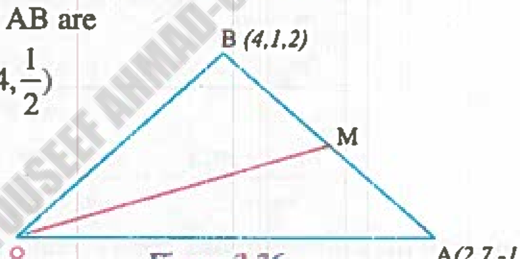


Figure 3.36

EXERCISE 3.2

1. If $a = 3i - 5j$ and $b = -2i + 3j$, then find
   (i) $a+2b$ (ii) $3a-2b$ (iii) $2(a-b)$
   (iv) $|a+b|$ (v) $|a|-|b|$ (vi) $\frac{|a|}{|b|}$
2. Find the unit vector having the same direction as the vector given below.
   (i) $3i$ (ii) $3i - 4j$ (iii) $i + j - 2k$ (iv) $\frac{\sqrt{3}}{2}i - \frac{1}{2}j$
3. If $r = i - 9j$, $a = i + 2j$ and $b = 5i - j$, determine the real numbers p and q such that $r = pa + qb$.
4. If $p = 2i - j$ and $q = xi + 3j$, then find the value of x such that $|p + q| = 5$.
5. Find the length of the vector $\overrightarrow{AB}$ from the point A(−3,5) to B (7,9). Also find a unit vector in the direction of $\overrightarrow{AB}$.
6. If ABCD is a parallelogram such that the coordinates of the vertices A, B and C are respectively given by (−2,−3), (1, 4) and (0, 5). Find the coordinates of the vertex D.

7. Find the components and the magnitude of $\overrightarrow{PQ}$
 i. P (–1,2), Q(2, –1).   ii. P (–2,1), Q(2,3).
 iii. P (–1,1,2), Q:(2, –1,3).   iv. P(2,4,6), Q(1, –2, 3).

8. Find the initial point P or the terminal point Q whichever is missing:
 i. $\overrightarrow{PQ}$ = [–2, 3], P (1, –2).   ii. $\overrightarrow{PQ}$ = [4, –5], Q(–1 , 1).
 iii. $\overrightarrow{PQ}$ = [–1,3, –2], P(2, –1, –3).   iv. $\overrightarrow{PQ}$ = [2, –3, –4], Q(3, – 1,4).

9. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=4\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$, find a vector of magnitude 6 units which is parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

10. Find the position vector of a point R which divides the line joining the points whose position vectors are $P(\hat{\imath}+2\hat{\jmath}-\hat{k})$ and $Q(-\hat{\imath}+\hat{\jmath}+\hat{k})$ in the ratio 2:1 internally and externally.

11. Find the position vectors of the point of division of the line segments joining
 (i) Point C with position vector $5j$ and point D with position vector $4i+j$ in the ratio 2 : 5 internally.
 (ii) Point E with position vector $2i-3j$ and point F with position vector $3i+2\hat{j}$ in the ratio 4 : 3 externally.

12. Find α, so that $\left|a\hat{\imath}+(a+1)\hat{\jmath}+2\hat{k}\right|=3$

13. If $\vec{u}=2\hat{i}+3\hat{j}+4\hat{k}$, $\vec{v}=-\hat{i}+3\hat{j}-\hat{k}$ and $\vec{w}=\hat{i}+6\hat{j}+z\hat{k}$ represent the sides of a triangle. Find the value of z.

14. The position vectors of the points A, B, C and D are $2\hat{i}-\hat{j}+\hat{k}$, $3\hat{i}+\hat{j}$, $2\hat{i}+4\hat{j}-2\hat{k}$ and $-\hat{i}-2\hat{j}+\hat{k}$ respectively. Show that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

3.5 Dot or Scalar Product

3.5.1 The **dot** or **scalar product** of two vectors ***a***, ***b*** denoted by ***a·b***, is defined as ***a·b*** = |***a***||***b***| cos θ where θ is the angle between the vectors ***a*** and ***b*** (Figure 3.37).

For example, if |*a*| = 2, |*b*| = 4, θ = 60°,

then $a\cdot b=3\times4\cos60° = 12\times\frac{1}{2}=6.$

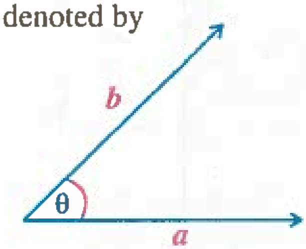


Figure 3.37

This will be negative if $\frac{\pi}{2}<\theta<\pi$ as $\cos\theta$ is negative, and |*a*|, |*b*| are always positive.

3.5.2 Immediate consequences of the definition of Dot Product

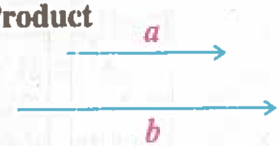


Figure 3.38

(i) Parallel vectors

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel but in the same direction as shown in (Figure 3.38), then $\theta = 0°$.

In this case $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos 0° = |\boldsymbol{a}||\boldsymbol{b}|$

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel but in opposite direction as shown in Figure (3.39), then $\theta=180°$. In this case $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos 180° = -|\boldsymbol{a}||\boldsymbol{b}|$

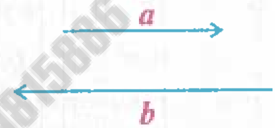


Figure 3:39

In the special case when $\boldsymbol{a} = \boldsymbol{b}$, then

$$\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}||\boldsymbol{a}|\cos 0° = |\boldsymbol{a}||\boldsymbol{a}| = |\boldsymbol{a}|^2$$

Hence $\quad |\boldsymbol{a}| = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$

(ii) Orthogonal vectors

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal vectors, then $\theta = 90°$ and $\cos 90° = 0$

$\therefore\ \boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos 90° = 0$

Hence the condition for orthogonality of two vectors is $\boldsymbol{a}\cdot\boldsymbol{b} = 0$

3.5.3 Scalar product of unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$

$i\cdot i = |i|\,|i| \cos 0° = 1$, $i\cdot j = |i|\,|j| \cos 90° = 0$

$j\cdot j = |j|\,|j| \cos 0° = 1$, $j\cdot k = |j|\,|k| \cos 90° = 0$

$k\cdot k = |k|\,|k| \cos 0° = 1$, $k\cdot i = |k|\,|i| \cos 90° = 0$

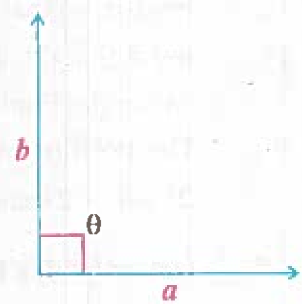


Figure 3.40

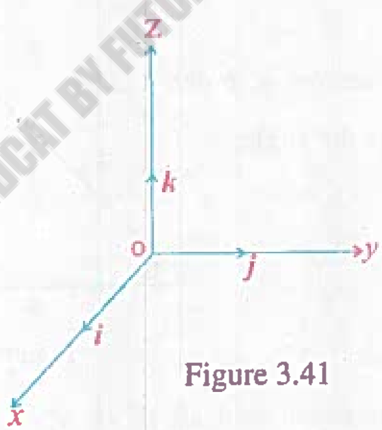


Figure 3.41

Remember

The dot product is always a number (scalar). We sometimes refer to it as the **scalar product** or **inner product**.

Thus, $\quad i\cdot i = j\cdot j = k\cdot k = 1 \quad and \quad i\cdot j = j\cdot k = k\cdot i = 0$

3.5.4 Expression of Dot Product in Terms of Components

Let $\boldsymbol{a} = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and $\boldsymbol{b} = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$ be two vectors in space. Then using the properties of dot product, we have

> **Remember**
>
> If $\boldsymbol{a} = x_1\boldsymbol{i} + y_1\boldsymbol{j}$ and $\boldsymbol{b} = x_2\boldsymbol{i} + y_2\boldsymbol{j}$ are vectors in the plane, then $\boldsymbol{a}\cdot\boldsymbol{b} = x_1x_2 + y_1y_2$

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b} &= (x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k})\cdot(x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k})\\ &= x_1x_2(\boldsymbol{i}\cdot\boldsymbol{i}) + x_1y_2(\boldsymbol{i}\cdot\boldsymbol{j}) + x_1z_2(\boldsymbol{i}\cdot\boldsymbol{k}) + y_1x_2(\boldsymbol{j}\cdot\boldsymbol{i}) + y_1y_2(\boldsymbol{j}\cdot\boldsymbol{j}) + y_1z_2(\boldsymbol{j}.\boldsymbol{k})\\ &\quad + z_1x_2(\boldsymbol{k}\cdot\boldsymbol{i}) + z_1y_2(\boldsymbol{k}\cdot\boldsymbol{j}) + z_1z_2(\boldsymbol{k}\cdot\boldsymbol{k}) = x_1x_2 + y_1y_2 + z_1z_2\end{aligned}$$

$$\therefore \quad \boldsymbol{a}\cdot\boldsymbol{b} = x_1x_2 + y_1y_2 + z_1z_2$$

Thus, dot product of two vectors is the sum of the product of their corresponding components.

Example 14: If $\boldsymbol{a} = 2i - 3j + 4k$ and $\boldsymbol{b} = i + 3j - 2k$, then find $\boldsymbol{a.b}$ in terms of their components.

Solution:

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b} &= (2i - 3j + 4k)\cdot(i + 3j - 2k)\\ &= (2)(1) + (-3)(3) + (4)(-2) = -15\end{aligned}$$

3.5.5 Commutative and Distributive Properties of Dot Product

Theorem: If $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are vectors and α any scalar, then

(a) Dot product is commutative i.e. $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$

(b) Dot product is distributive over vector addition i.e. $\boldsymbol{a}\cdot(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{c}$

Proof:

(a) Let $\boldsymbol{a} = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and $\boldsymbol{b} = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$

Using the properties of dot product and scalars, we have

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b} &= (x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}).(x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k})\\ &= x_1x_2 + y_1y_2 + z_1z_2 = x_2x_1 + y_2y_1 + z_2z_1\\ &= \boldsymbol{b}\cdot\boldsymbol{a}\end{aligned}$$

Thus, $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$

(b) Let $\boldsymbol{c} = x_3\boldsymbol{i} + y_3\boldsymbol{j} + z_3\boldsymbol{k}$, then

$$\begin{aligned}\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) &= (x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}).[(x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}) + (x_3\boldsymbol{i} + y_3\boldsymbol{j} + z_3\boldsymbol{k})]\\ &= (x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}).[(x_2 + x_3)\boldsymbol{i} + (y_2 + y_3)\boldsymbol{j} + (z_2 + z_3)\boldsymbol{k}]\\ &= x_1(x_2 + x_3) + y_1(y_2 + y_3) + z_1(z_2 + z_3)\\ &= x_1x_2 + x_1x_3 + y_1y_2 + y_1y_3 + z_1z_2 + z_1z_3\\ &= (x_1x_2 + y_1y_2 + z_1z_2) + (x_1x_3 + y_1y_3 + z_1z_3) = \boldsymbol{a.b} + \boldsymbol{a.c}\end{aligned}$$

Thus, $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{c}$

3.5.6 Direction Angles and Direction Cosines of Vectors

Let $r = xi + yj + zk$ be a non–zero vector. Let α, β and γ be the angles which the vector r makes with the positive directions of the coordinate axes where each of these angles lies between 0 and π i.e. $0 \le \alpha, \beta, \gamma \le \pi$.

The angles α, β and γ are called the **direction angles** of the vector r (see **Figure 3.42**).

Referring to the figure, we have three right triangles OAP, OBP and OCP. Then

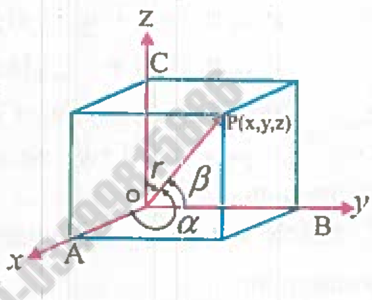


Figure 3.42

$$\cos\alpha = \frac{x}{|r|} = \frac{x}{\sqrt{x^2+y^2+z^2}} \text{ in right triangle OAP}$$

$$\cos\beta = \frac{y}{|r|} = \frac{y}{\sqrt{x^2+y^2+z^2}} \text{ in right triangle OBP}$$

$$\cos\gamma = \frac{z}{|r|} = \frac{z}{\sqrt{x^2+y^2+z^2}} \text{ in right triangle OCP}$$

The numbers $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **direction cosines** of the vector r. The direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are usually denoted by l, m and n respectively.

Theorem: If α, β and γ are the direction angles of a vector r, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Proof: By the definition of direction cosines of the vector r, we have

$$\cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}}, \cos\beta = \frac{y}{\sqrt{x^2+y^2+z^2}} \text{ and } \cos\gamma = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{x^2+y^2+z^2} + \frac{y^2}{x^2+y^2+z^2} + \frac{z^2}{x^2+y^2+z^2}$$

$$= \frac{x^2+y^2+z^2}{x^2+y^2+z^2} = 1$$

Using symbols ℓ,m and n, we may write the above result in the form $\ell^2 + m^2 + n^2 = 1$

Example 15: Find the direction cosines of the vector from P(4,8,–3) to Q(–1,6,2)

Solution: We know that for any two points P and Q we have $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

Here $\overrightarrow{OQ} = -i + 6j + 2k$, $\overrightarrow{OP} = 4i + 8j - 3k$

$\overrightarrow{PQ} = (-i + 6j + 2k) - (4i + 8j - 3k) = -5i - 2j + 5k$

Since $|\overrightarrow{PQ}| = \sqrt{x^2+y^2+z^2}$

So $|\overrightarrow{PQ}| = \sqrt{(-5)^2+(-2)^2+(5)^2} = \sqrt{54} = 3\sqrt{6}$

Hence direction cosines of the vector $\overrightarrow{PQ}$ are

$$\cos\alpha = \frac{x}{|\overrightarrow{PQ}|} = \frac{-5}{3\sqrt{6}},\ \cos\beta = \frac{y}{|\overrightarrow{PQ}|} = \frac{-2}{3\sqrt{6}} \text{ and } \cos\gamma = \frac{z}{|\overrightarrow{PQ}|} = \frac{5}{3\sqrt{6}}$$

3.5.7 Direction Numbers or Direction Ratios

The position vector $\overrightarrow{OP}$ of the point P(x,y,z) in term of unit vectors i,j and k is given as

$$\overrightarrow{OP} = r = xi+yj+zk$$

If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of r, and p is a positive constant, then the numbers $p\cos\alpha$, $p\cos\beta$ and $p\cos\gamma$ are called the **direction numbers** or **direction ratios** of the vector r. The direction numbers are used to specify the direction of the vector r.

Since $x = |r|\cos\alpha$, $y = |r|\cos\beta$ and $z = |r|\cos\gamma$ where $|r|$ is the length of the vector r, so x, y and z are direction numbers of the vector r. Therefore the coordinates of P(x,y,z) may be written as $(|r|\cos\alpha, |r|\cos\beta, |r|\cos\gamma)$

Hence $\overrightarrow{OP} = r = |r|\cos\alpha\, i + |r|\cos\beta\, j + |r|\cos r\, k$

$\Rightarrow \quad \overrightarrow{OP} = |r|(\cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k)$ or

$\overrightarrow{OP} = |r|\ (li+mj+nk)$

Did You Know ?

From the above, we obtain the components of r from the direction cosines multiplying by $|r|$.
Conversely, dividing the components by $|r|$ gives the direction cosines.

Example 16: Find the direction numbers and direction cosines of the point P(2,–3,6).

Solution: The direction numbers are 2, –3, 6.

Since $\overrightarrow{OP} = r = 2i - 3j + 6k$, $|\overrightarrow{OP}| = 7$, therefore the direction cosines are

$$\ell = \frac{x}{|\overrightarrow{OP}|} = \frac{2}{7},\ m = \frac{y}{|\overrightarrow{OP}|} = \frac{-3}{7} \text{ and } n = \frac{z}{|\overrightarrow{OP}|} = \frac{6}{7}$$

Example 17: Find the coordinates of P, if $\overrightarrow{OP}$ is of length 6 units in the direction of $\overrightarrow{OR}$ where R is the point (2, –1, 4)

Solution: We have $\overrightarrow{OR} = 2i - j + 4k \quad \therefore \quad |\overrightarrow{OR}| = \sqrt{21}$

The direction cosines of $\overrightarrow{OR}$ are

$$\ell = \frac{2}{\sqrt{21}},\ m = \frac{-1}{\sqrt{21}},\ n = \frac{4}{\sqrt{21}}$$

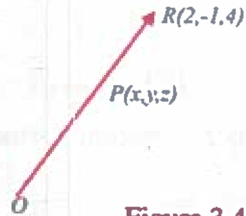


Figure 3.43

The coordinates of P are $(|r|\ell, |r|m, |r|n)$

where $|r| = |\overrightarrow{OP}| = 6$.

Therefore $|r|\,\ell = \frac{12}{\sqrt{21}}$, $|r|m = \frac{-6}{\sqrt{21}}$, $|r|n = \frac{24}{\sqrt{21}}$

Hence the coordinates of P are $\left(\frac{12}{\sqrt{21}}, \frac{-6}{\sqrt{21}}, \frac{24}{\sqrt{21}}\right)$.

Example 18: A vector v has inclination 60° to $\overrightarrow{ox}$, 45° to $\overrightarrow{oy}$.

Find its inclination to $\overrightarrow{oz}$. If $|v| = 12$, express v as $xi + yj + zk$.

Solution: Here $l = \cos 60° = \frac{1}{2}$, $m = \cos 45° = \frac{1}{\sqrt{2}}$

Let $n = \cos\gamma$, where γ is inclination to $\overrightarrow{oz}$.

Since $l^2 + m^2 + n^2 = 1$

So $n^2 = 1 - l^2 - m^2 \Rightarrow n^2 = 1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4}$

$\Rightarrow\ n = \pm\frac{1}{2}$ $\qquad\therefore\ \cos\gamma = \pm\frac{1}{2}$

This shows that v is inclined to $\overrightarrow{oz}$ either at 60° or 120°.

Now ℓi, mj, nk are components of $\hat{v}$, so $\hat{v} = \frac{1}{2}i + \frac{1}{\sqrt{2}}j + \frac{1}{2}k$

But $v = |v|\hat{v} = 12\left(\frac{1}{2}i + \frac{1}{\sqrt{2}}j + \frac{1}{2}k\right) = 6i + 6\sqrt{2}\,j + 6k$

3.5.8 Angle between two Vectors

One use of the dot product is to calculate the angle between two vectors.

(i) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the two vectors. Then by definition of dot product

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta \text{ where } 0 \le \theta \le \pi$$

$$\therefore \cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

i.e. the cosine of the angle between two vectors is their dot product divided by the product of their moduli.

(ii) if $\boldsymbol{a} = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and $\boldsymbol{b} = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$

$\boldsymbol{a}\cdot\boldsymbol{b} = x_1x_2 + y_1y_2 + z_1z_2$

$|a| = \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $|b| = \sqrt{x_2^2 + y_2^2 + z_2^2}$ since by (i) above

$$\cos\theta = \frac{a\cdot b}{|a||b|} \quad \therefore \cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}}$$

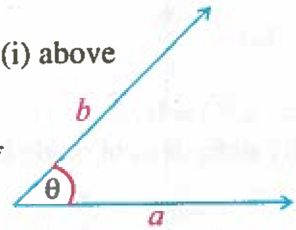


Figure 3.44

(iii) $a.b = |a||b| \cos\theta \qquad \cos\theta = \dfrac{a\cdot b}{|a||b|} = \dfrac{a}{|a|}\cdot\dfrac{b}{|b|} = \hat{a}.\hat{b}$

Example 19: Find the angle between the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ where $\overrightarrow{OP} = 2i + j$,

$\overrightarrow{OQ} = -3i + 2j$

Solution: Let θ be the angle between the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$

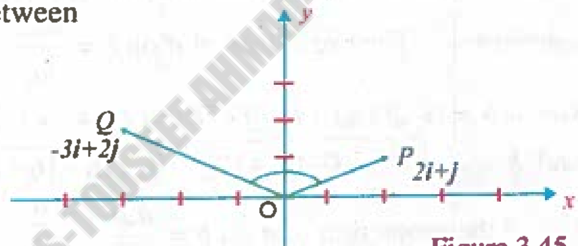


Figure 3.45

Then $\cos\theta = \dfrac{\overrightarrow{OP}.\overrightarrow{OQ}}{|\overrightarrow{OP}||\overrightarrow{OQ}|}$

$= \dfrac{(-3i+2j).(2i+j)}{\sqrt{3^2+2^2}\sqrt{2^2+1^2}}$

$\Rightarrow \cos\theta = \dfrac{-6+2}{\sqrt{13}\sqrt{5}} = -0.4961$

$\Rightarrow \theta = 119.74°$

Example 20: Find the value of t such that the vectors $2i - j + 2k$ and $3i + 2tj$ are orthogonal.

Solution: Let $\boldsymbol{a} = 2i - j + 2k$ and $\boldsymbol{b} = 3i + 2tj$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, then $\boldsymbol{a}\cdot\boldsymbol{b} = 0$

$\therefore\ 2(3) + (-1)(2t) + 2(0) = 0 \quad \Rightarrow \quad -2t = -6 \text{ or } t = 3$

3.5.9 Projection of one Vector on another

Let a and b be two vectors and θ be the angle between them as shown in (Figure 3.46) , $0 \le \theta \le \pi$

$\overrightarrow{AC}$ is perpendicular to $\overrightarrow{OB}$. Then $\overrightarrow{OC}$ is called the **projection** of a on b.

From Δ OCA, we have

$$\frac{|\overrightarrow{OC}|}{|\overrightarrow{OA}|} = \cos\ \theta$$

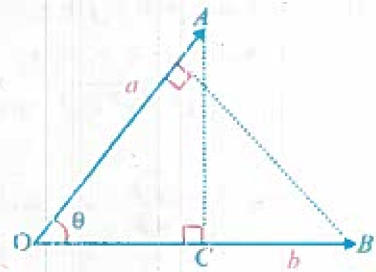


Figure 3.46

$$\Rightarrow |\overrightarrow{OC}| = |\overrightarrow{OA}|\cos\theta = |a|\cos\theta \qquad (1)$$

By definition of angle between vectors

$$\cos\theta = \frac{a.b}{|a||b|} \qquad (2)$$

Using (1) and (2), we have $|\overrightarrow{OC}| = \dfrac{a\cdot b}{|b|}$

This gives that the projection of ***a*** on ***b*** is $\dfrac{\vec{a}\cdot\vec{b}}{|b|}$. Similarly the projection of ***b*** on ***a*** is $\dfrac{a.b}{|a|}$

Example 21: Find the projection of the vector $\boldsymbol{a} = i-2j+k$ to the vector $\boldsymbol{b} = 4i-4j+7k$.

Solution: The projection of ***a*** on ***b*** $= \dfrac{a.b}{|b|}$

Now $a\cdot b = (i-2j+k).(4i-4j+7k) = (1)(4)+(-2)(-4)+(1)(7) = 19$

And $|b| = \sqrt{(4)^2+(-4)^2+(7)^2} = \sqrt{16+16+49} = \sqrt{81} = 9$

$\therefore$ the projection of ***a*** on ***b*** $= \dfrac{a.b}{|b|} = \dfrac{19}{9}$

3.5.10 Work done by a constant force

If a constant force F acts on an object during any interval of time and the object undergoes a displacement S, then the work done on the object by the force F is defined as

$$W = \vec{F}.\vec{S}$$

or $W = FS\ Cos\ \theta$, where θ is the angle between the directions of $\vec{F}$ and $\vec{S}$, as in (Figure 3.47).

Figure 3.47

Example 22: Find the work done in moving an object along a vector $9i - j + k$ if the applied force is $3i + 2j + k$.

Solution: Here $\vec{F} = 3i + 2j + k$

$\vec{S} = 9i - j + k$

$$\therefore \quad W = \vec{F}\cdot\vec{S} = (3i+2j+k)\cdot(9i-j+k)$$
$$= 3(9) + 3(-1) + 1(1)$$
$$= 27 - 2 + 1$$
$$= 26$$

Hence work done = 26 units

EXERCISE 3.3

1. If $a = 3i + 4j - k$, $b = i - j + 3k$ and $c = 2i + j - 5k$ then find
   (i) $a \cdot b$ (ii) $a \cdot c$ (iii) $a \cdot (b+c)$
   (iv) $(2a+3b) \cdot c$ (v) $(a-b) \cdot c$

2. Write a unit vector in the direction of the sum of the vectors
   $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$

3. Find the angles between the following pairs of vectors:
   (i) $i - j + k$, $-i + j + 2k$ (ii) $3i + 4j$, $2j - 5k$ (iii) $2i - 3k$, $i + j + k$

4. Show that $i + 7j + 3k$ is perpendicular to both $i - j + 2k$ and $2i + j - 3k$.

5. Let $a = i + 2j + k$ and $b = 2i + j - k$. Find a vector that is orthogonal to both a and b.

6. Let $a = i + 3j - 4k$ and $b = 2i - 3j + 5k$. Find the value of m so that $a + mb$ is orthogonal to (i) a (ii) b

7. Given the vectors $\vec{a}$ and $\vec{b}$ as follows:
   (i) $a = -\frac{3}{2}j + \frac{4}{5}k$, $b = i - 2j - 2k$ (ii) $a = -3i + j + 2k$, $b = -i - j + 5k$
   Find in each case the projection of **a** on **b** and of **b** on **a**.

8. What is the cosine of the angle which the vector $\sqrt{2}\hat{i} + \hat{j} + \hat{k}$ makes with y-axis?

9. A force $F = 2i + 3j + k$ acts through a displacement $S = 2i + j - k$.
   Find the work done.

10. Find the work done by the force $\vec{F} = 2i + 3j + k$ in the displacement of an object from a point A(–2,1,2) to the point B(5,0,3).

11. (i) Show that the vectors $3i - 2j + k$, $i - 3j + 5k$ and $2i + j - 4k$ form a right triangle. (ii) Show that the set of points $P = (1,0,1)$, $Q = (1,1,1)$ and $R = (1,1,0)$ form a right isosceles triangle.

12. Prove that the angle in a semicircle is a right angle.

13. Prove that perpendicular bisectors of the sides of a triangle are concurrent.

3.6 The Cross or Vector Product of two Vectors

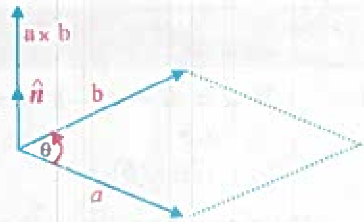


Figure 3.48(a)

In section 3.5 we noticed that dot product of two vectors in plane or in space gives a scalar. However, in this section we shall see that there is another product known as cross or vector product, which gives the result as vector in three dimensional space.

3.6.1 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non–zero vectors. The **cross** or **vector product** of $\boldsymbol{a}$ and $\boldsymbol{b}$, denoted as $\boldsymbol{a} \times \boldsymbol{b}$, is defined by

$$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta\, \hat{n}$$

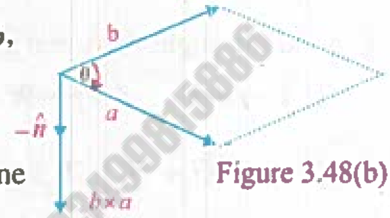


Figure 3.48(b)

where $\hat{n}$ is a unit vector perpendicular to the plane determined by $\boldsymbol{a}$ and $\boldsymbol{b}$ (See Figure 3.48 (a))

The direction of $\hat{n}$ is determined by the right hand rule
"Join the tails of $\boldsymbol{a}$, $\boldsymbol{b}$, stretch the fingers of your right hand along the direction of first vector $\boldsymbol{a}$ and curl them towards the second vector $\boldsymbol{b}$ through smaller angle θ between $\boldsymbol{a}$ and $\boldsymbol{b}$ ($0<\theta<180°$), then the erected thumb will show the direction of $\hat{n}$ or $\boldsymbol{a} \times \boldsymbol{b}$."

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are as shown in (Figure 3.48 (a)). and the plane containing $\boldsymbol{a}$, $\boldsymbol{b}$ represents upper surface of a table then $\boldsymbol{a} \times \boldsymbol{b}$ is directed above the table.

Clearly, the direction of $\boldsymbol{b} \times \boldsymbol{a}$ by stretching fingers along $\boldsymbol{b}$ and curling towards $\boldsymbol{a}$ gives the direction of the thumb of right hand downwards (under the table) direction from the plane (see Figure 3.48(b)).

Hence $\quad \boldsymbol{b} \times \boldsymbol{a} = -|\boldsymbol{b}|\,|\boldsymbol{a}| \sin\theta\, \hat{n}$

where $\hat{n}$ is a unit vector perpendicular to the plane directed upward.

In Figure 3.48(b) $\boldsymbol{b} \times \boldsymbol{a}$ is the scalar multiple of $-\hat{n}$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors, then the length of $\boldsymbol{a} \times \boldsymbol{b}$ is given by $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$

3.6.2 Immediate consequences of the definition of Cross Product

(i) Since $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$, hence vector product is not commutative i.e. $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{b} \times \boldsymbol{a}$.

(ii) **Parallel Vectors.** If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel but in the opposite direction as shown in Figure 3.49(a), then $\theta = 180°$.

Figure 3.49(a)

In this case $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin 180°\, \hat{n} = 0$
($\because \sin 180° = 0$)

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel but in the same direction as shown in Figure 3.49(b), then $\theta=0°$

In this case $\boldsymbol{a}\times\boldsymbol{b}=|\boldsymbol{a}|\,|\boldsymbol{b}|\sin 0°\,\hat{n}=0 \quad (\because \sin 0°=0)$

Hence in either case $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{0}$

If $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{0}$, then either at least one of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ is zero or $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

In particular $\boldsymbol{a}\times\boldsymbol{0}=\boldsymbol{0}$ for all vectors $\boldsymbol{a}$.

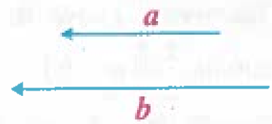


Figure 3.49(b)

3.6.3 Expressing Cross Product in terms of components

Let $\boldsymbol{a}=x_1i+y_1j+z_1k$ and $\boldsymbol{b}=x_2i+y_2j+z_2k$ be two vectors in space. Then using the properties of cross product, we have

$$a\times b=(x_1i+y_1j+z_1k)\times(x_2i+y_2j+z_2k)$$

$$=x_1x_2(i\times i)+x_1y_2(i\times j)+x_1z_2(i\times k)+y_1x_2(j\times i)+y_1y_2(j\times j)+y_1z_2(j\times k)$$
$$+z_1x_2(k\times i)+z_1y_2(k\times j)+z_1z_2(k\times k)$$

$$=x_1x_2(0)+x_1y_2(k)+x_1z_2(-j)+y_1x_2(-k)+y_1y_2(0)+y_1z_2(i)+z_1x_2(j)+z_1y_2(-i)+z_1z_2(0)$$

$$=(y_1z_2-z_1y_2)i-(x_1z_2-z_1x_2)j+(x_1y_2-y_1x_2)k \qquad (1)$$

The expansion of 3×3 determinant

$$\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=(y_1z_2-z_1y_2)i-(x_1z_2-z_1x_2)j+(x_1y_2-y_1x_2)k \qquad (2)$$

From (1) and (2), we have $\quad \boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$

3.6.4 Application to Geometry

Theorem: Prove that the magnitude of $\boldsymbol{a}\times\boldsymbol{b}$ represents the area of a parallelogram with adjacent sides $\boldsymbol{a}$ and $\boldsymbol{b}$.

Proof: Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non–zero vectors representing the two adjacent sides of the parallelogram and θ be the angle between them as shown in Figure 3.50. We know from geometry that

Area of parallelogram = base × altitude

$= |\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta=|\boldsymbol{a}\times\boldsymbol{b}|$

Thus Area of parallelogram $=|\boldsymbol{a}\times\boldsymbol{b}|$

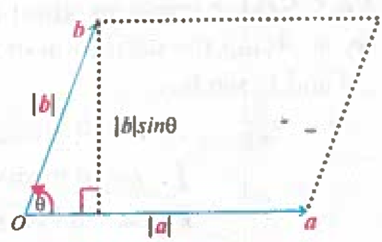


Figure 3.50

Theorem: Prove that the area of a triangle equals $\frac{1}{2}$ $|\boldsymbol{a} \times \boldsymbol{b}|$.

Proof: From Figure 3.51, we have that

area of triangle = $\frac{1}{2}$ (area of parallelogram).

Figure 3.51

By above theorem

Area of parallelogram = $|\boldsymbol{a} \times \boldsymbol{b}|$ $\quad \therefore \quad$ Area of triangle = $\frac{1}{2}$ $|\boldsymbol{a} \times \boldsymbol{b}|$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors along the two adjacent sides of the triangle.

Example 23: Find the area of the triangle whose vertices are A(2,2,0), B(−1,0,2) and C(0,4,3).

Solution: Let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ be the adjacent sides of the parallelogram determined, so the required area of the triangle is half the area of the parallelogram, that is

$$\text{Area of the triangle} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

Since $\overrightarrow{AB} = (-1,0,2) - (2,2,0) = (-3,-2,2)$ and

$\overrightarrow{AC} = (0,4,3) - (2,2,0) = (-2,2,3)$,

$$\text{so } \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix} = -10i + 5j - 10k$$

$$\Rightarrow \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-10)^2 + (5)^2 + (-10)^2} = \sqrt{225} = 15$$

$$\therefore \quad \text{Area of the triangle} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{15}{2}$$

3.6.5. Scalar triple product of *i*, *j* and *k*

By applying the definition of cross product to unit vectors *i*, *j* and *k*, we have

(a) $i \times i = |i|\,|i| \sin 0° \hat{n} = 0$

$j \times j = |j|\,|j| \sin 0° \hat{n} = 0$

$k \times k = |k|\,|k| \sin 0° \hat{n} = 0$

(b) $i \times j = |i|\,|j| \sin 90° k = k$

$j \times k = |j|\,|k| \sin 90° i = i$

$k \times i = |k|\,|i| \sin 90° j = j$

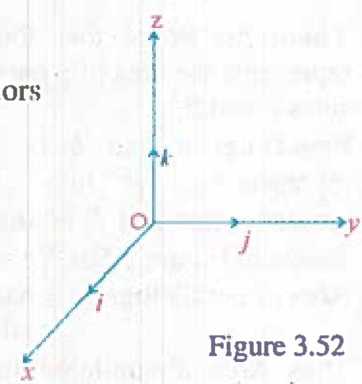


Figure 3.52

(c) $j \times i = -(i \times j) = -k$

$k \times j = -(j \times k) = -i$

$i \times k = -(k \times i) = -j$

Thus

$i \times i = j \times j = k \times k = 0$

$i \times j = k, j \times k = i, k \times i = j$

$j \times i = -k, k \times j = -i, i \times k = -j$

For convenience we arrange unit vectors i, j, k in clockwise order as shown in Figure 3.53. Then the cross product of any two consecutive vectors is the remaining third vector with a plus sign or a minus sign according as the order of the product is clockwise or anticlockwise.

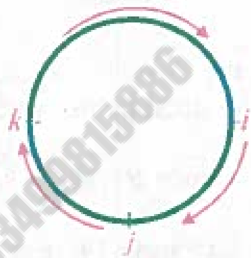


Figure 3.53

3.6.6 Anticommutative Property

Theorem: If a, b are vectors, then

$$a \times b = -b \times a$$

Proof:

This property has already been proved geometrically. Analytically we prove it as follows.

Let $a = x_1 i + y_1 j + z_1 k$, $b = x_2 i + y_2 j + z_2 k$.

$$a \times b = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = -\begin{vmatrix} i & j & k \\ x_2 & y_2 & z_2 \\ x_1 & y_1 & z_1 \end{vmatrix} \text{ (by interchanging the rows of the determinant)}$$

$$= -b \times a$$

Thus $a \times b = -b \times a$

If $a = 0$ or $b = 0$ or $\sin\theta = 0$, then clearly $a \times b = 0$

3.6.7 Distributive Property

Theorem: If a, b and c are vectors, then

(i) $(a + b) \times c = a \times c + b \times c$ (ii) $a \times (b + c) = a \times b + a \times c$

Proof:

(i) Let $a = x_1 i + y_1 j + z_1 k$, $b = x_2 i + y_2 j + z_2 k$ and $c = x_3 i + y_3 j + z_3 k$, then $a+b=(x_1+x_2)i+(y_1+y_2)j+(z_1+z_2)k$ and so

$$(a+b) \times c = \begin{vmatrix} i & j & k \\ x_1+x_2 & y_1+y_2 & z_1+z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = a \times c + b \times c$$

Thus $(a + b) \times c = a \times c + b \times c$

(ii) Proof is similar to (i) above

3.6.8 Angle between two vectors

One use of the cross product is to calculate the angle between two vectors.

(i) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the two vectors. Then by definition of cross product

$$|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta \text{ where } 0\le\theta\le\pi$$

$$\therefore\ \sin\theta=\frac{|a\times b|}{|a||b|}$$ i.e. the sine of the angle between the two vectors is the modulus of their cross product divided by the product of their moduli.

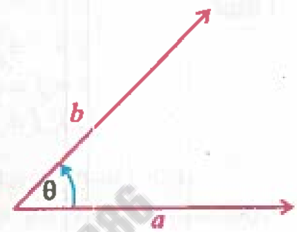


Figure 3.54

Hence $\theta=\sin^{-1}\dfrac{|a\times b|}{|a||b|}$

Example 24: For the vectors $\boldsymbol{a} = 2i + 5j + 3k$, $\boldsymbol{b} = 3i + 3j + 6k$, and $\boldsymbol{c}=2i+7j+k$, find

(i) $(a-b)\times(c-a)$

(ii) a unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$

(iii) $\sin\theta$ where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

Solution: (i) $a-b = (2i+5j+3k)-(3i+3j+6k) = -i+2j-3k$

$c-a = (2i+7j+4k)-(2i+5j+3k) = 2j+k$

$$\therefore (a-b)\times(c-a) = \begin{vmatrix} i & j & k \\ -1 & 2 & -3 \\ 0 & 2 & 1 \end{vmatrix} = (2+6)\,i-(-1-0)j+(-2-0)k = 8i+j-2k$$

(ii) Let $\hat{n}$ be the required unit vector orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$, then

$$\hat{n}=\frac{a\times b}{|a\times b|} \qquad (1)$$

$$a\times b = \begin{vmatrix} i & j & k \\ 2 & 5 & 3 \\ 3 & 3 & 6 \end{vmatrix} = (30-9)i-(12-9)j+(6-15)k = 21i-3j-9k,$$

$$|a\times b| = \sqrt{(21)^2+(-3)^2+(-9)^2} = \sqrt{531} = 3\sqrt{59}$$

Putting in (1), we have

$$\hat{n} = \frac{21i-3j-9k}{3\sqrt{59}} = \frac{7i-j-3k}{\sqrt{59}}$$

(iii) Since $|a\times b| = |a|\,|b|\sin\theta$ where θ is the angle between $\boldsymbol{a}$, $\boldsymbol{b}$, we have

$$\sin\theta = \frac{|a\times b|}{|a||b|}\ \therefore\ \sin\theta = \frac{3\sqrt{59}}{\sqrt{38}\sqrt{54}} = \frac{3\sqrt{59}}{\sqrt{38\times54}} = \frac{3\sqrt{59}}{3\sqrt{228}} \Rightarrow \sin\theta = \frac{\sqrt{59}}{\sqrt{228}}$$

3.6.9 Moment of Force

The moment M of a force F about a point P is defined as the product $M = |F|\ d$, where d is the (perpendicular) distance between P and the line of action L of F as shown in (Figure 3.55)

If r is the vector from P to any point Q on L, then

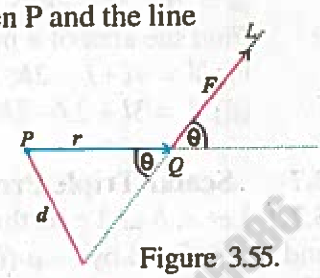


Figure 3.55.

$$d = |r|\ \sin\theta$$

$$\therefore \quad M = |F|d = |r||F|\sin\theta$$

Since θ is the angle between r and F, so $M = r\times F$

The vector $M = r\times F$ is called the **moment vector** or **vector moment** of $\overline{F}$ about Q.

Example 25: Find the moment about a point A(2,1,1) of the force $\overline{F} = 7i + 4j - 3k$ applied at (1,–2,3)

Solution: If r is the position vector of the point P of application relative to the point A about which the moment is calculated, then moment M is given by $M = r\times F$

where $r = \overline{AP} = (2i+j+k) - (i-2j+3k) = -(i+3j-2k)$. Hence,

$$M = r\times F = \begin{vmatrix} i & j & k \\ 1 & 3 & -2 \\ 7 & 4 & -3 \end{vmatrix} = -[(-9+8)i + (-14+3)j + (4-21)k] = i + 11j + 17k$$

EXERCISE 3.4

1. Find the following cross products.
   (i) $j\times(2j+3k)$ (ii) $(2i-3j)\times k$ (iii) $(2i-3j+5k)\times(6i+2j-3k)$
2. Show in two different ways that the vectors $\vec{a}$ and $\vec{b}$ are parallel:
   (i) $\vec{a} = -i+2j-3k$, $\vec{b} = 2i-4j+6k$ (ii) $\vec{a} = 3i+6j-9k$, $\vec{b} = i+2j-3k$
3. Find a unit vector that is orthogonal to the given two vectors:
   (i) $\vec{a} = i-2j+3k$, $\vec{b} = 2i+j-k$ (ii) $\vec{a} = 3i-j+6k$, $\vec{b} = i+4j+k$
4. If $\vec{a} = 3i-6j+5k$, $\vec{b} = 2i-j+4k$, $\vec{c} = i+j-k$, compute
   (i) $\vec{a}\times\vec{b}$ (ii) $\vec{b}\times\vec{c}$ (iii) $(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})$
5. Use the vector product to compute the area of the triangle with the given vertices: (i) P: (–2,–3), Q: (3,2) , R: (–1,–8)
   (ii) P: (–2, –1,3), Q: (1,2, –1), R: (4,3, –3)
6. A force $F = 3i-2j+5k$ acts on a particle at (1, –2, 2). Find the moment or torque of the force about (i) the origin; (ii) the point (1,2,1).
7. If $A+B+C=0$, show that $A\times B = B\times C = C\times A$.

8. (i) Find a unit vector perpendicular to both $\vec{a} = i + j + 2k$, and $\vec{b} = -2i + j - 3k$
   (ii) Find a vector of magnitude 10 and perpendicular to both $\vec{a} = 2i - 3j + 4k$, $\vec{b} = 4i - 2j - 4k$.
9. Find the area of a parallelogram whose diagonals are:
   (i) $\vec{a} = 4i + j - 2k$ and $\vec{b} = -2i + 3j + 4k$
   (ii) $\vec{a} = 3i + 2j - 2k$ and $\vec{b} = i - 3j + 4k$

3.7 Scalar Triple Product of Vectors

3.7.1 Let *a, b* and *c* be three vectors. The **scalar triple product** of the vectors *a, b* and *c* is defined by $a\cdot(b \times c)$ or $(a \times b)\cdot c$

The use of parenthesis with $a \times b$ is not important, as the only other alternative given to the expression $a \times b\cdot c$, namely $a \times (b\cdot c)$ is meaningless. The scalar triple product $a\cdot b \times c$ is usually denoted by $[a\ b\ c]$.

3.7.2 Expression of Scalar Triple Product in Terms of Components

Let $a = x_1 i + y_1 j + z_1 k$, $b = x_2 i + y_2 j + z_2 k$ and $c = x_3 i + y_3 j + z_3 k$ be vectors, then

$$b \times c = \begin{vmatrix} i & j & k \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Rightarrow b \times c = (y_2 z_3 - y_3 z_2)i - (x_2 z_3 - x_3 z_2)j + (x_2 y_3 - x_3 y_2)k$$

$$\therefore \quad a\cdot(b \times c) = x_1(y_2 z_3 - y_3 z_2) - y_1(x_2 z_3 - x_3 z_2) + z_1(x_2 y_3 - x_3 y_2)$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Thus $$a\cdot(b \times c) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

which is called the **determinantal form** for scalar triple product of vectors *a, b* and *c*.

Theorem: For any vectors *a, b* and *c*, $a\cdot(b \times c) = b\cdot(c \times a) = c\cdot(a \times b)$

Proof: Let $a = x_1 i + y_1 j + z_1 k$, $b = x_2 i + y_2 j + z_2 k$ and $c = x_3 i + y_3 i + z_3 k$, then by determinantal form for scalar triple product of vectors *a, b* and *c*, we have

$$a\cdot(b \times c) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad (1)$$

Similarly $b\cdot(c\times a) = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \end{vmatrix} = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_2 & y_2 & z_2 \end{vmatrix}$

$$\Rightarrow \quad b\cdot(c\times a) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad (2)$$

and $\quad c\cdot(a\times b) = \begin{vmatrix} x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_2 & y_2 & z_2 \end{vmatrix}$

$$\Rightarrow \quad c\cdot(a\times b) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad (3)$$

From (1), (2) and (3), we have

$$a\cdot(b\times c) = b\cdot(c\times a) = c\cdot(a\times b)$$

By virtue of above theorem $[a\ b\ c] = [b\ c\ a] = [c\ a\ b]$

3.7.3. Scalar triple product of i, j and k

Theorem: Let i, j and k be the unit vectors. Prove that

(a) $\quad i\cdot j\times k = j\cdot k\times i = k\cdot i\times j = 1$ and (b) $\quad i\cdot k\times j = j\cdot i\times k = k\cdot j\times i = -1$

Proof: The proof is simple, so it is left for students.

3.7.4 Dot and cross are inter-changeable in scalar triple product

Theorem: The positions of dot and cross in the scalar triple product can be interchanged.

Proof: Let $a = x_1i+y_1j+z_1k$, and $b=x_2i+y_2j+z_2k$ and $c=x_3i+y_3j+z_3k$ be any three vectors. Then

$$a\cdot(b\times c) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad (1)$$

By definition

$$a\times b = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2-z_1y_2)i-(x_1z_2-z_1x_2)j+(x_1y_2-y_1x_2)k$$

$$\therefore \quad (a\times b)\cdot c = (y_1z_2-z_1y_2)x_3-(x_1z_2-z_1x_2)y_3+(x_1y_2-y_1x_2)z_3 \qquad (2)$$

$$= \begin{vmatrix} x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

From (1) and (2), we have $\quad \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$

This shows that the position of dot and cross in the scalar triple product can be interchanged.

3.7.5 (a) The Volume of the Parallelepiped

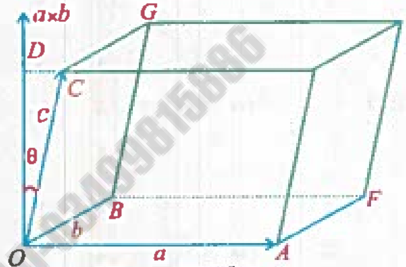


Figure 3.56

Let us consider the parallelepiped with ***a, b*** and ***c*** as co–terminal edges are shown in (Figure 3.56).

Then $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{c} = \overrightarrow{OC}$

Let $\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{d}$. Then by definition of cross product ***d*** is perpendicular to the plane containing ***a*** and ***b*** and geometrically represents the area of the parallelogram OAFB given by $|\boldsymbol{a}\times\boldsymbol{b}|$. The parallelogram is regarded as base for the parallelepiped. If θ is the angle between the vectors ***d*** and ***c***, then $|\overrightarrow{OD}| = |\boldsymbol{c}|\cos\theta$ being the projection of ***c*** on ***d*** represents the height of the parallelepiped. Then from elementary geometry, we know that the volume ***v*** of the parallelepiped is the area of the base multiplied by height.

Hence volume of parallelepiped $=$ (Area of parallelogram) (Height)

$$\Rightarrow v = |\boldsymbol{a}\times\boldsymbol{b}|\,|\boldsymbol{c}|\cos\theta$$

$$\Rightarrow v = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$$

The scalar triple product will be positive if θ is acute and ***c*** lies on the same side of the plane which contains ***a*** and ***b***.

As $|\boldsymbol{b}\times\boldsymbol{c}|$ represents the area of the other side OCGB of parallelepiped, hence

$$v = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$$

Therefore $\quad v = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$

This shows that $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ or $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$ is the volume of the parallelepiped with ***a, b,*** and ***c*** as the co–terminal edges.

3.7.5 (b) Volume of Tetrahedron A tetrahedron is determined by three edge vectors ***a, b, c*** as shown in (Figure 3.57).

The volume of a tetrahedron with ***a, b, c*** as its co–terminal edges is given by

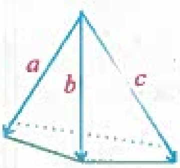


Figure 3.57

$$v = \frac{1}{6}\,[\boldsymbol{a}\;\boldsymbol{b}\;\boldsymbol{c}] = \frac{1}{6}\,\{\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})\}$$

3.7.6 Properties of Scalar Triple Product

(i) $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$ being the volume of a parallelepiped with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as co–terminal edges, hence the evaluation of the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$ gives the volume of the parallelepiped as discussed earlier.

(ii) If two of the three vectors are equal, then the value of the scalar triple product is zero because for any two identical rows, the determinant vanishes.

(iii) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$ if and only if the three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar.

Example 26: Find the volume of the parallelepiped determined by

$\mathbf{a} = 2i + 3k,\quad \mathbf{b} = 6j - 2k,\quad$ and $\mathbf{c} = -3i + 3j$

Solution: Let v be the volume of the given parallelepiped.

Then
$$V = \mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 6 & -2 \\ -3 & 3 & 0 \end{vmatrix} = 2(0+6) - 0(0-6) + 3(0+18)$$
$$= 12 - 0 + 54 = 66$$

Example 27: Find the volume of tetrahedron with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as adjacent edges where

$\mathbf{a} = i + 2k,\ \ \mathbf{b} = 4i + 6j + 2k\ \$ and $\ \mathbf{c} = 3i + 3j - 6k$

Solution: Let V be the volume of tetrahedron.

Then
$$V = \frac{1}{6}\begin{vmatrix} 1 & 0 & 2 \\ 4 & 6 & 2 \\ 3 & 3 & -6 \end{vmatrix} = \frac{1}{6}\left[(-36-6) - 0(-24-6) + 2(12-18)\right]$$
$$= \frac{1}{6}(-42-12) = \frac{-54}{6} = 9$$

We ignore the minus sign, because volume is always non–negative.

> **Did You Know ?**
>
> Two or more vectors are said to be coplanar if they lie in the same plane or parallel to the same plane otherwise non–coplanar. Non–coplanar vectors lie in three–dimensional space.

Example 29: Show that the points A(4, –2,1), B(5,1,6), C(2, 2, –5), D(3, 5, 0) are coplanar.

Solution:

Let $\mathbf{a} = \overrightarrow{AB} = (5-4)i + (1+2)j + (6-1)k = i + 3j + 5k$

$\mathbf{b} = \overrightarrow{BC} = (2-5)i + (2-1)j + (-5-6)k = -3i + j - 11k$

$\mathbf{c} = \overrightarrow{CD} = (3-2)i + (5-2)j + (0+5)k = i + 3j + 5k$

The four points are coplanar if the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ are coplanar. We have

$$[a\ b\ c] = \begin{vmatrix} 1 & 3 & 5 \\ -3 & 1 & -11 \\ 1 & 3 & 5 \end{vmatrix} = 1(5+33)-3(-15+11)-5(-9-1) = 38+12-5=0$$

Hence the four points are coplanar.

EXERCISE 3.5

1. Find $\vec{a}.\left(\vec{b}\times\vec{c}\right)$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$
2. Find the volume of the parallelepiped whose edges are represented by $a = 3i + j - k$, $b = 2i - 3j + k$, $c = i - 3j - 4k$
3. For the vectors $a = 3i + 2k$, $b = i + 2j + k$, $c = -j + 4k$ verify that $a \cdot b \times c = b \cdot c \times a = c \cdot a \times b$ but $a \cdot b \times c = -\ c \times b \cdot a$
4. Verify that the triple product of $i - j$, $j - k$, and $k - i$ is zero.
5. Let $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$. Find $a \times b$ and prove that
   (i) $a \times b$ is orthogonal to both a and b (use dot product) (ii) Find $(a \times b)^2$
   (iii) Find $(a \cdot b)^2$, $|a|^2$, $|b|^2$
   (iv) Show that $|a \times b|^2 = (a \cdot a)\ (b \cdot b) - (a \cdot b)^2$
6. Do the points (4,–2,1), (5, 1, 6), (2, 2, –5) and (3, 5, 0) lie in a plane?
7. For what values of c the following vectors are coplanar?
   (i) $u = i + 2j + 3k$, $v = 2i - 3j + 4k$, $w = 3i + j + ck$
   (ii) $u = i + j - k$, $v = i - 2j + k$, $w = ci + j - ck$
   (iii) $u = i + j + 2k$, $v = 2i + 3j + k$, $w = ci + 2j + 6k$
8. Find the volume of tetrahedron with the following
   (i) Vectors as coterminous edges $a = i + 2j + 3k$, $b = 4i + 5j + 6k$, $c = 7j + 8k$
   (ii) Points A(2,3,1), B(–1,–2,0), C(0,2,–5), D(0, 1,–2) as vertices.
9. (i) Write the value of $(\hat{\imath} \times \hat{\jmath}).\,\hat{k} + \hat{\imath}.\,\hat{\jmath}$ (ii) Write the value of $(\hat{k} \times \hat{\jmath}).\,\hat{\imath} + \hat{\jmath}.\hat{k}$

REVIEW EXERCISE 3

1. Choose the correct option.

(i) The value of $\vec{i}\cdot(\vec{j}\times\vec{k}) + \vec{j}\cdot(\vec{i}\times\vec{k}) + \vec{k}\cdot(\vec{i}\times\vec{j})$

(a) 0 (b) –1 (c) 1 (d) 3

(ii) The vector $3\bar{i} + 5\bar{j} + 2\bar{k}$, $2\bar{i} - 3\bar{j} - 5\bar{k}$ and $5\bar{i} + 2\bar{j} - 3\bar{k}$ form the sides of a triangle which is

(a) Equilateral (b) isosceles, but not right-angled

(c) Right-angled, but not isosceles (d) right-angled and isosceles

(iii) The two vectors $\vec{a}=2\vec{i}+\vec{j}+3\vec{k}, \vec{b}=4\vec{i}-\lambda\vec{j}+6\vec{k}$ are parallel if $\lambda=$

(a) 2 (b) −3 (c) 3 (d) −2

(iv) If $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$, then

(a) $\vec{a}$ is parallel to $\vec{b}$ (b) $\vec{a}\perp\vec{b}$ (c) $|\vec{a}|=|\vec{b}|$ (d) None of these

(v) The projection of the vector $2\vec{i}+3\vec{j}-2\vec{k}$ on the vector $\vec{i}+2\vec{j}+3\vec{k}$ is

(a) $\frac{1}{\sqrt{14}}$ (b) $\frac{2}{\sqrt{14}}$ (c) $\frac{3}{\sqrt{14}}$ (d) None of these

(vi) Find non-zero scalars α, β for which $\alpha(\vec{a}+2\vec{b})-\beta\vec{a}+(4\vec{b}-\vec{a})=0$ for all vectors $\vec{a}$ and $\vec{b}$.

(a) $\alpha=-2, \beta=-3$ (b) $\alpha=2, \beta=-3$

(c) $\alpha=1, \beta=-3$ (d) $\alpha=-2, \beta=3$

(vii) If a, b, c are position vectors of the vertices of a Δ ABC, then

$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=$

(a) 0 (b) 2a (c) 2b (d) 3c

(viii) If θ be the angle between any two vector $\vec{a}$ and $\vec{b}$, then $|\vec{a}\cdot\vec{b}|=|\vec{a}\times\vec{b}|$, when θ is equal to

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

2. Find λ and μ if $(\vec{i}+3\vec{j}+9\vec{k})\times(3\vec{i}-\lambda\vec{j}+\mu\vec{k})=\vec{0}$.

3. If $\vec{a}=9\vec{i}-\vec{j}+\vec{k}$ and $\vec{b}=2\vec{i}-2\vec{j}-\vec{k}$, then find a unit vector parallel to the vector $\vec{a}+\vec{b}$.

4. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, find $(\vec{r}\times\hat{i}).(\vec{r}\times\hat{j})+xy$.

5. If $\vec{a}=7\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$.

6. Find λ, if the vectors $\vec{a}=\hat{i}+3\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}-\hat{k}$ and $\vec{c}=\lambda\hat{j}+3\hat{k}$ are coplanar.

7. Vector $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=\sqrt{3}$, $|\vec{b}|=\frac{2}{3}$ and $(\vec{a}\times\vec{b})$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}$.

8. Find the area of a triangle whose vertices are (0,0,2), (−1,3,2), (1,0,4).

9. Find the area of the parallelogram with vertices A(1,2,−3), B(5,8,1), C (4,−2,2),D(0,−8,−2).

10. Prove that in any triangle ABC

(i) $a^2=b^2+c^2-2bc\cos A$ (Cosine Law)

(ii) $a=b\cos C+c\cos B$ (Projection Law)

UNIT 4

SEQUENCES AND SERIES

$$a_n = a_1 + (n-1)\,d$$

- a_n: n^{th} term in the sequence
- a_1: 1^{st} term in the sequence
- n: number of terms in the sequence
- d: common difference

STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Define a sequence (progression) and its terms.
- Know that a sequence can be constructed from a formula or an inductive definition.
- Recognize triangle, factorial and pascal sequences.
- Define an arithmetic sequence.
- Find the nth or general term of an arithmetic sequence.
- Solve problems involving arithmetic sequence.
- Know arithmetic mean between two numbers.
- Insert n arithmetic means between two numbers.
- Define an arithmetic series.
- Establish the formula to find the sum to n terms of an arithmetic series.
- Show that sum of n arithmetic means between two numbers is equal to n times their arithmetic mean.
- Solve real life problems involving arithmetic series.
- Define a geometric sequence.
- Find the nth or general term of a geometric sequence.
- Solve problems involving geometric sequence.
- Know geometric mean between two numbers.
- Insert n geometric means between two numbers.
- Define a geometric series.
- Find the sum of n terms of a geometric series.
- Find the sum of an infinite geometric series.
- Convert the recurring decimal into an equivalent common fraction.
- Solve real life problems involving geometric series.
- Recognize a harmonic sequence.
- Find nth term of harmonic sequence.
- Define a harmonic mean.
- Insert n harmonic means between two numbers.

4.1 Introduction

In practical life you must have observed many things follow a certain pattern, such as the petals of a sunflower, the holes of a honeycomb, the grains on a maize cob, the spirals on a pineapple on a pipe cone etc.

In our day-to-day life, we see patterns of geometric figures on clothes, pictures, posters etc. They make the learners motivated to form such new patterns.

Number patterns are faced by learners in their study. Number patterns play an important role in the field of mathematics. Let us study the following number patterns

(i) 2, 4, 6, 8, 10,... (ii) $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, \ldots$ (iii) 10, 7, 4, 1, −2,... (iv) 2,4,8,16,32,...

(v) $4, \frac{1}{2}, \frac{1}{16}, \frac{1}{28}, \ldots$ (vi) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ (vii) 1, 11, 111,1111, 11111,...

It is an interesting study to find whether some specific names have been given to some of the above number patterns and the methods of finding some next terms of the given patterns.

Observing various patterns various sequences were defined to solve various summation problems.

Among various sequences A.P. (Arithmetic progression), G.P. (Geometric progression) and H.P (Harmonic progression) are most common.

Idea on A.P. was given by mathematician Carl Friedrich Gauss, who, as a young boy, stunned his teacher by adding up $1 + 2 + 3 + \ldots\ldots + 99 + 100$ within a few minutes. Here's how he did it.

He realized that adding the first and last numbers, 1 and 100, gives, 101 and adding the second, and second last numbers, 2 and 99, gives 101, as well as $3 + 98 = 101$ and so on, Thus he concluded that there are 50 sets of 101. So the sum of the series is $50(1 + 100) = 5050$.

4.1.1. Sequence

A **sequence** is a function whose domain is the set of positive integers. The numbers in the range of a sequence are real numbers, called terms of the sequence.

4.1.2 Construction of a sequence from a formula (inductive definition)

Let f be a function defined by

$$f(n) = 2n, \; n \in \{1, 2, 3, \ldots\}$$

then $f(1) = 2$, the first term

$f(2) = 4$, the second term

$f(3) = 6$, the third term ....

Thus the required sequence is 2, 4, 6, …

Remember

A Sequence is also called a progression

In sequences, instead of using a symbol such as $f(n)$ for the nth term (usually called the general term) which denotes the number that corresponds to a given integer n, it is customary to use the symbol a_n for f(n). When the nth term of a sequence is known then we denote the entire sequence by the symbol $\{a_n\}$, where $a_1, a_2, a_3, \ldots$ are the first term, the second term, the third term of the sequence $\{a_n\}$ and so on. Since the order among the positive integers induce the ordering among the corresponding terms of the sequence, this clearly shows that the ordering has a vital role in the definition of a sequence, so we can also define a sequence as follows.

A **sequence** is a collection of numbers arranged in particular order.

The sequence 1,1,2,3,5,8,...can be written as $x_1 = x_2 = 1$, $x_n = x_{n-1} + x_{n-2}$, $n > 2$, $n \in N$.

This sequence of numbers is called the Fibonacci sequence. Some sequences may not be described by any rule 2, 3, 5, 7, 11, 13, 17,... the formula for a_n, the nth prime number has not been found yet.

Remember

A sequence may be described by specifying first few terms and a formula (or a set of formulae) giving a relation between successive terms. Such a formula is called RECURSIVE FORMULA (or RECURRENCE RELATION).

Example 1: Write the first four terms a_1, a_2, a_3, and a_4 of each sequence, where $a_n = f(a)$.

(a) $f(n) = 2n - 5$ (b) $f(n) = 4(2)^{n-1}$

(c) $f(n) = (-1)^n\left(\frac{n}{n+1}\right)$

Solution:

a) Since $f(n) = 2n - 5$

$$a_1 = f(1) = 2(1) - 5 = -3$$

$$a_2 = f(2) = 2(2) - 5 = -1$$

In a similar manner, $a_3 = f(3) = 1$ and $a_4 = f(4) = 3$.

b) Since $f(n) = 4(2)^{n-1}$

$a_1 = f(1) = 4(2)^{1-1} = 4$

Similarly, $a_2 = 8$, $a_3 = 16$ and $a_4 = 32$.

c) $a_1 = f(1) = (-1)^1 (1/1+1) = -1/2$,

$$a_2 = f(2) = (-1)^2 (2/2+1) = 2/3,$$

$$a_3 = f(3) = (-1)^3 (3/3+1) = -3/4,$$

$$a_4 = f(4) = (-1)^4 (4/4+1) = 4/5.$$

Remember

A sequence is said to be finite if there is a first and last term otherwise it is said to be infinite.

Did You Know

The factor $(-1)^n$ causes the terms of the sequence to alternate signs

Example 2: Find the first four terms of the recursive sequence that is defined by
$a_n = 2a_{n-1} + 1; \quad a_1 = 3.$

Solution: The sequence is defined recursively, so we must find the terms in order.

$a_1 = 3$

$a_2 = 2a_1 + 1 = 2(3) + 1 = 7$

$a_3 = 2a_2 + 1 = 2(7) + 1 = 15$

$a_4 = 2a_3 + 1 = 2(15) + 1 = 31$

The first four terms are 3,7,15, and 31.

Did You Know

There is no unique representation of a general term of a sequence. It is not always possible to determine the general term of a sequence.

4.1.3 Some special Sequences

Some well-known sequences are given in the following example.

Example 3: Write down the first five terms of each recursively defined sequence.

(a) $t_1 = 1\ ,\ t_{n+1} = t_n + (n+1)\ ,\ n = 1, 2, 3, \ldots$

(b) $f_0 = 1\ , f_{r+1} = (r+1) f_r\ ,\ r = 0, 1, 2, 3, \ldots$

(c) $p_0 = 1,\ \ p_{r+1} = \dfrac{4-r}{r+1} p_r,\ \ r = 0, 1, 2, 3, \ldots$

Solution: (a)

$t_1 = 1$

$t_2 = t_1 + 2 = 1 + 2 = 3$

$t_3 = t_2 + 3 = 1 + 2 + 3 = 6$

$t_4 = t_3 + 4 = 1 + 2 + 3 + 4 = 10$

$t_5 = t_4 + 5 = 1 + 2 + 3 + 4 + 5 = 15$

(b)

$f_0 = 1$

$f_1 = 1.f_0 = 1 \times 1 = 1$

$f_2 = 2.f_1 = 2 \times 1 = 2$

$f_3 = 3.f_2 = 3 \times 2 \times 1 = 6$

$f_4 = 4.f_3 = 4 \times 3 \times 2 \times 1 = 24$

This sequence is so important that it has its own special notation, r !, read as 'r factorial'. It is defined as: 0! = 1 and (r + 1)! = (r + 1) × r!, r = 0, 1, 2, . . .

(c) $p_0 = 1$

$$p_1 = \frac{4}{1} p_0 = \left(\frac{4}{1}\right)(1) = 4 \qquad p_2 = \frac{3}{2} p_1 = \left(\frac{3}{2}\right)(4) = 6$$

$$p_3 = \frac{2}{3} p_2 = \left(\frac{2}{3}\right)(6) = 4 \qquad p_4 = \frac{1}{4} p_3 = \left(\frac{1}{4}\right)(4) = 1$$

The sequences given in example 3(a), 3(b) and 3(c) are special type of sequences, called the triangle number sequence, the factorial sequence and Pascal sequence respectively. These sequences play an important role in the expansion of binomial expressions like $(x + y)^n$.
The complete Pascal sequence in 3(c) is

$$1, 4, 6, 4, 1, 0, 0, 0, 0, \ldots$$

This is only one of the families of Pascal sequences. Due to its importance, it has a special notation,

$$\binom{4}{r}, \; r = 0, 1, 2, 3, \ldots$$

For example $\binom{4}{0} = 1$, $\binom{4}{1} = 4$, $\binom{4}{2} = 6$, $\binom{4}{3} = 4$, $\binom{4}{4} = 1$, $\binom{4}{5} = 0$ and so on.

Obviously, different Pascal sequences will have different multiplying factor.

The general definition of a Pascal sequence is:

$$\binom{n}{0} = 1 \text{ and } \binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}, \quad r = 0, 1, 2, 3, \ldots$$

We obtain the following Pascal sequences for $n = 0, 1, 2, 3, 4, \ldots$ by using the above general definition:

n = 0: 1, 0, 0, 0, 0, 0, . . .
n = 1: 1, 1, 0, 0, 0, 0, . . .
n = 2: 1, 2, 1, 0, 0, 0, . . .
n = 3: 1, 3, 3, 1, 0, 0, . . .
n = 4: 1, 4, 6, 4, 1, 0, . . .

from which we extract the well-known triangle, called

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Did You Know

Summation Notation

Summation notation is used to write series effectively. The symbol $\sum$, the uppercase Greek letter sigma, indicates a sum.

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \dots\dots + a_n$$

The letter k is called the index of summation. The numbers 1 and n represent the sub-scripts of the first and last terms in the series. They are called the lower limit and upper limit of the summation, respectively.

Practice

Using summation notation
Evaluate each series

(a) $\sum_{k=1}^{n} k^2$

(b) $\sum_{k=1}^{n} 5$

(c) $\sum_{k=1}^{n} (2k-5)$

Remember

Sequences and Series

A sequence is an ordered list, whereas a series is a sum of the terms of a sequence.
For example.
1, 3, 5, 7, 9, 11, 13, 15 is a sequence, and
1+3+5+7+9+11+13+15 a series.

Example 4: Find the sum $\sum_{k=1}^{4} k^2(k-2)$

Solution:

$$\sum_{k=1}^{4} k^2(k-2) = 1^2(1-2)+2^2(2-2)+3^2(3-2)+4^2(4-2)$$

$$=(-1)+0+(9)+(32) \quad =40$$

Example 5: Find the sum $\sum_{k=1}^{10} c$, c is constant

Solution:

$$\sum_{k=1}^{10} c = c+c+c+\dots+c = 10c$$

EXERCISE 4.1

1. Classify the following into finite and infinite sequences.
   (i) $2,4,6,8,\ldots,50$ (ii) $1,0,1,0,1,\ldots$
   (iii) $\ldots,-4,0,4,8,\ldots,60$ (iv) $1,-\frac{1}{3},\frac{1}{9},-\frac{1}{27},\ldots,-\frac{1}{2187}$
2. Find the first four terms of a sequence with the given general terms:
   (i) $\frac{n(n+1)}{2}$ (ii) $(-1)^{n-1}2^{n+1}$
   (iii) $\left(\frac{1}{3}\right)^{n}$ (iv) $\frac{n(n-1)(n-2)}{6}$
3. Write down the nth term of each sequence as suggested by the pattern.
   (i) $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\ldots$ (ii) $2,-4,6,-8,10,\ldots$ (iii) $1,-1,1,-1,\ldots$
4. Write down the first five terms of each sequence defined recursively.
   (i) $a_1=3, a_{n+1}=5-a_n$ (ii) $a_1=3, a_{n+1}=\frac{a_n}{n}$
5. Write each of the following series in expanded form.
   (i) $\sum_{j=1}^{6}(2j-3)$ (ii) $\sum_{k=1}^{5}(-1)^{k}2^{k-1}$
   (iii) $\sum_{j=1}^{\infty}\frac{1}{2^{j}}$ (iv) $\sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^{k}$
6. Find the Pascal sequences for: (i) n=5 (ii) n=6 (iii) n=8 by using its general recursive definition.

4.2 Arithmetic Sequence (A.P)

4.2.1. Numbers are said to be in **Arithmetic Sequence (A.S)** ***or Arithmetic Progression (A.P)*** when its terms increase or decrease by a common difference. Thus each of the following ***sequence*** forms an Arithmetic Progression:

3, 7, 11, 15, ……………………….

8, 2, –4, –10, ……………………..

a, a+d, a+2d, a+3d,………………

The common difference is found by subtracting any term of the series from that which follows *it*. In the first of the above examples the common difference is 4; in the second it is –6; in the third it is d.

4.2.2 The nth term of an Arithmetic Sequence

We find a formula for the nth term of an arithmetic sequence. Let a_1 be its first term and d be its common difference. Then consecutive terms of the sequence are given by

$$a_1 = a_1 + 0.d = a_1 + (1-1)d$$
$$a_2 = a_1 + d = a_1 + 1\cdot d = a_1 + (2-1)d$$
$$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2\cdot d = a_1 + (3-1)d$$
$$a_4 = a_3 + d = (a_1 + 2\cdot d) + d = a_1 + 3\cdot d = a_1 + (4-1)d$$
$$a_5 = a_4 + d = (a_1 + 3\cdot d) + d = a_1 + 4\cdot d = a_1 + (5-1)d$$
$$\vdots$$
$$\Rightarrow a_n = a_1 + (n-1)d$$

Example 7: Find the 15th term of the arithmetic sequence whose first three terms are 20, 16.5 and 13.

Solution: Here $a_1 = 20, d = 16.5 - 20 = -3.5$ and $n = 15$. Substituting these values in the formula: $a_n = a_1 + (n-1)d$

We obtain, $a_{15} = 20 + (15-1)(-3.5) = 20 - 49 = -29$

If any two terms of an Arithmetic *sequence* be given, the series can be completely determined; for the data furnish two simultaneous equations, the solution of which will give the first term and the common difference.

Example 8: The 8th term of an arithmetic sequence is 75 and the 20th term is 39. Find the first term and the common difference. Give a recursive formula for the sequence.

Solution: We know that $a_n = a_1 + (n-1)d$

then $a_8 = a_1 + 7d = 75$ (i)

and $a_{20} = a_1 + 19d = 39$ (ii)

Subtracting (ii) from (i), we obtain

$$-12d = 36 \Rightarrow d = -3$$

From (i) we get $a_1 + 7(-3) = 75$ or $a_1 = 96$

Since $a_n = a_1 + (n-1)d = 96 + (n-1)(-3) = 99 - 3n$

$$a_{n+1} = 99 - 3(n+1) = 99 - 3n - 3 = 96 - 3n = (99 - 3n) - 3 = a_n - 3$$

$\therefore$ $a_{n+1} = a_n - 3$ is the required recursive formula for the given arithmetic sequence.

4.3 Arithmetic Mean (A.M)

4.3.1 When three numbers are in Arithmetic Progression, the middle one is said to be the arithmetic mean of the other two.

Thus arithmetic mean of two numbers a and b is $\frac{a+b}{2}$, where a and b are called the extremes. Mathematically, it is derived in the following way:

Let A be the arithmetic mean between two numbers a and b, then a, A, b, form an arithmetic sequence. By definition, we have

$$A-a=b-A$$

$$2A=a+b$$

Hence $$A=\frac{a+b}{2}$$

Thus the arithmetic mean of two numbers is equal to one–half of their sum.

Example 9: Find the arithmetic mean of $\sqrt{2}-3$ and $\sqrt{2}+3$

Solution: Here $a=\sqrt{2}-3$, $b=\sqrt{2}+3$

$$A=\frac{a+b}{2}=\frac{\sqrt{2}-3+\sqrt{2}+3}{2}=\sqrt{2}$$

Between two given numbers it is always possible to insert any number of terms such that the whole series thus formed shall be an A. P.; the terms thus inserted are called the arithmetic means.

4.3.2 Inserting n Arithmetic Means (A.Ms)

Let $A_1, A_2, ..., A_n$ be n A.Ms between a and b then $a, A_1, A_2, ...A_n, b$ form a finite arithmetic sequence of $n+2$ terms, that is:

$$a_{n+2}=b$$

$a+(n+2-1)d=b$, where d is the common difference

$$(n+1)d=b-a$$

$$\therefore d=\frac{b-a}{n+1}$$

Thus $$A_1=a+d=a+\frac{b-a}{n+1}$$

$$A_2=a+2d=a+2\left(\frac{b-a}{n+1}\right)$$

$$A_3=a+3\left(\frac{b-a}{n+1}\right)$$

Similarly $$A_4=a+4\left(\frac{b-a}{n+1}\right)$$

$\vdots$

$$A_n = a + n\left(\frac{b-a}{n+1}\right)$$

which are the required n A.Ms between a and b. Thus, $A_1, A_2, \ldots, A_n$ are real numbers such that $a, A_1, A_2, \ldots A_n, b$ is an arithmetic sequence, then $A_1, A_2, \ldots, A_n$ are called the n arithmetic means between the numbers a and b. The process of determining these numbers is referred to as inserting n arithmetic means between a and b.

Example 10: Insert three arithmetic means between 2 and 9.

Solution: Let A_1, A_2 and A_3 be the arithmetic means between 2 and 9 such that $2, A_1, A_2, A_3, 9$ forms a finite arithmetic sequence of 5 terms with $a = 2$, $b = 9$. Let d be the common difference, then $a_5 = b$ gives that

$$a + 4d = 9 \Rightarrow 2 + 4d = 9$$

$4d = 7 \Rightarrow d = \frac{7}{4}$ Thus the three arithmetic means are

$$A_1 = a + d = 2 + \frac{7}{4} = \frac{15}{4}$$

$$A_2 = a + 2d = 2 + 2\left(\frac{7}{4}\right) = \frac{11}{2}$$

$$A_3 = a + 3d = 2 + 3\left(\frac{7}{4}\right) = \frac{29}{4}$$

EXERCISE 4.2

1. Find the 15th term of the arithmetic sequence 2, 5, 8, ...
2. The 1^{st} term of an arithmetic sequence is 8 and the 21^{st} term is 108. Find the 7th term.
3. Find the number of terms in the arithmetic progression 6, 9, 12, ..., 78.
4. The nth term of a sequence is given by $a_n = 2n + 7$. Show that it is an A.P. Also, find its 7^{th} term.
5. Show that the sequence log a, log(ab), log(ab^2), log(ab^3), ... is an A.P. Find its nth term.
6. Find the value of 'k' if 2k+7, 6k−2, 8k−4 are in A.P. Also find the sequence.

7. If $a_6 + a_4 = 6$ and $a_6 - a_4 = \frac{2}{3}$, find the arithmetic sequence.

8. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P. then prove $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

9. A ball rolling up an incline covered 24m during the first second, 21m during the second, 18m during the third second. Find how many meters it covered in the eighth second?

10. The population of a town is decreasing by 500 inhabitants each year If its population in 1960 was 20135, what was its population in 1970?

11. Ahmad and Akram can climb 1000 feet in the first hour and 100 feet in each succeeding hour. When will they reach the top of a 5400 feet hill?

12. A man earned $3500 the first year he worked. If he received a raise of $750 at the end of each year for 20 years, what was his salary during his twenty first year of work?

13. Find the arithmetic mean between the given numbers:

(i) 12, 18 (ii) $\frac{1}{3}, \frac{1}{4}$ (iii) $-6, -216$ (iv) $(a+b)^2, (a-b)^2$

14. Insert: (i) Three arithmetic means between 6 and 41.

(ii) Four arithmetic means between 17 and 32.

15. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the arithmetic mean between a and b?

16. Insert five arithmetic means between 5 and 8 and show that their sum is five times the arithmetic mean between 5 and 8.

17. There are n arithmetic means between 5 and 32 such that the ratio of the 3rd and 7th means is 7:13, find the value of n.

4.4 Arithmetic Series

4.4.1 As we know that associated with every sequence is a series, the indicated sum of the terms of the sequence. If the sequence happens to be the arithmetic sequence, then the associated series is called the arithmetic series.

Let $\{a_n\}$ be the arithmetic sequence then the series

$$a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k \text{ is called the arithmetic series.}$$

The arithmetic series in the general form or standard form is given as:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + [a_1 + (n-1)d] = \sum_{k=1}^{n} [a_1 + (k-1)d] \quad (1)$$

where S_n denotes the sum of the first n terms of the arithmetic series.

4.4.2 Sum of first n terms of an Arithmetic Series

The next result gives a formula for finding the sum of the first n terms of an arithmetic sequence.

Theorem: For an arithmetic sequence $\{a_n\}$, the sum S_n of the first n terms is given by

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$
$$= \frac{n}{2}(a_1 + a_n)$$

Proof: The sum of the first n terms of an arithmetic sequence is denoted by S_n.

Let $\quad S_n = a_1 + a_2 + a_3 + \ldots\ldots + a_n$

Since d is the common difference between terms, S_n can be written forward and backward as follows.

Forward: Start with the first term, a_1. Keep adding d.

Backward: Start with the last term, a_n. Keep subtracting d.

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + a_n$$
$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + a_1$$
$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n)$$

Add the two equations.

$$= n \text{ sums of } (a_1 + a_n)$$
$$2S_n = n(a_1 + a_n)$$

Solve for S_n, dividing both sides by 2.

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$a_n = a_1 + (n-1)d,$$
$$\therefore S_n = \frac{n}{2}\{a_1 + a_1 + (n-1)d\}.$$
$$\therefore S_n = \frac{n}{2}\{2a_1 + (n-1)d\}.$$

Example 11:
Finding the sum of a finite arithmetic series. Use a formula to find the sum of the arithmetic series $2 + 4 + 6 + 8 + \ldots\ldots\ldots + 100$.

Solution: The series $2 + 4 + 6 + 8 + \ldots\ldots + 100$ has n=50 terms with $a_1 = 2$ and $a_{50} = 100$. We can use the formula $S_n = \frac{n}{2}(a_1 + a_n)$ to find its sum.

$$S_{50} = \frac{50}{2}(2 + 100) = 2550$$

We can also use the formula $\quad S_n = \frac{n}{2}\{2a_1 + (n-1)d\}.$

$$S_{50} = 50/2\,(2(2) + (50-1)\,2) = 2550$$

Example 12: The sum of an arithmetic series with 15 terms is 285. If $a_{15} = 40$, find a_1

Solution: To find a_1, we apply the sum formula

$$S_n = \frac{n}{2}(a_1 + a_n) \quad \text{with } n = 15 \text{ and } a_{15} = 40$$

$$\frac{15}{2}(a_1 + 40) = 285$$

$$15(a_1 + 40) = 570 \quad \text{Multiply by 2.}$$

$$(a_1 + 40) = 38 \quad \text{Divide by 15.}$$

$$a_1 = -2 \quad \text{Subtract 40}$$

Example 13: The first term of a series is 5, the last 45 and the sum 400. Find the numebr of terms, and the common difference.

Solution: If n be the number of terms, then from

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$400 = \frac{n}{2}(5 + 45);$$

Hence $\quad n = 16.$

If d be the common difference

$$45 = \text{the } 16^{th} \text{ term} = 5 + 15d;$$

Hence $\quad d = 2\frac{2}{3}.$

Example 14: Find the sum of the first 200 positive odd integers.

Solution: Since the positive odd integers:

1,3,5,…,2n–1,… form an arithmetic sequence with

$$a_1 = 1, d = 2, n = 200 \text{ then } a_n = a_1 + (n-1)d$$

$$= 1 + (200 - 1)(2) = 399$$

$$\therefore S_n = \frac{n}{2}(a_1 + a_n) = \frac{200}{2}(1 + 399) = \frac{200}{2}(400) = 40000$$

Example 15: Find the 18^{th} term and the sum of the 18 terms of the arithmetic sequence: – 8, – 3, 2, 7, …

Solution: Since we are given that:

– 8, – 3, 2, 7, … is an arithmetic sequence.

Then $a_1 = -8, d = 5$ and n = 18. We have to find a_{18} and S_{18}

Since $\quad a_n = a_1 + (n - 1)\, d$

$\quad a_{18} = -8 + 17\,(5)$; *putting values of a_1 & d*

$$= 77$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{18} = \frac{18}{2}(-8+77) \text{ putting values of } a_1 \text{ \& } a_n$$

$$= 9\,(69) = 621$$

Example 16: The 10th term of an arithmetic sequence is 32 and the 18^{th} term is 48, what is the sum of the first twelve terms?

Solution: Let a_1 be the first term, d be the common difference and n be the number of terms of the given arithmetic sequence.

Then $a_{10} = 32, \quad a_{18} = 48$

$$a_1 + (10-1)d = 32, \quad a_1 + (18-1)d = 48$$

$$a_1 + 9d = 32 \qquad \text{(i)}$$

$$a_1 + 17d = 48 \qquad \text{(ii)}$$

Subtracting (i) from (ii) we obtain

$$8d = 16$$

$$d = 2$$

(i) gives that $a_1 + 18 = 32$

$$a_1 = 14$$

Now $S_n = \frac{n}{2}\{2a_1 + (n-1)d\}$

$$S_{12} = \frac{12}{2}\{2(14) + 11(2)\} \quad = 6\{28+22\} = 300$$

Example 17: Find the sum of all the integers lying between 100 and 600 that end in 5.

Solution: The integers lying between 100 and 600 that end in 5 are

105,115, 125, ..., 595

which form an arithmetic sequence with

$$a_1 = 105, d = 10, a_n = 595$$

then $a_n = a_1 + (n-1)d$

$$595 = 105 + 10n - 10$$

$$10n = 595 - 105 + 10$$

$$= 500$$

$$n = 50$$

Since $S_n = \frac{n}{2}(a_1 + a_n)$

Which gives $S_{50} = \frac{50}{2}(105+595)$

$$= 25(700)$$
$$= 17500$$

4.4.3 Relation of A.M of two numbers with n A.Ms between them

Theorem: The sum of n arithmetic means between two numbers is equal to n times their arithmetic mean.

Proof: Let there be n arithmetic means between a and b such that

$$a, a+d, a+2d, ..., a+nd, b$$

forms an arithmetic sequence with n+2 terms. Then

$$a+(a+d)+(a+2d)+...+(a+nd)+b = \frac{n+2}{2}[2a+(n+2-1)d]$$
$$= \frac{n+2}{2}[a+\{a+(n+1)d\}]$$
$$= \frac{n+2}{2}[a+b] \quad , b = a+(n+1)d$$
$$(a+d)+(a+2d)+...+(a+nd) = \frac{n+2}{2}(a+b)-(a+b)$$
$$= (a+b)\left[\frac{n+2}{2}-1\right]$$
$$= \text{n}\left(\frac{a+b}{2}\right)$$

Thus the sum of n arithmetic means = n (arithmetic mean)

4.4.4 Real life problems involving arithmetic series

Example 18: Finding the sum of a finite arithmetic series

A person has a starting annual salary of Rs.300,000 and receives a 1500 raise each year.

(a) Calculate the total amount earned over 9 years.

(b) Verify this value using a calculator.

Solution: (a) Using $S_n = \frac{n}{2}\{2a_1+(n-1)d\}$,

$$S_{10} = \frac{10}{2}\{2\times 300000+(10-1)1500\} = 3,067,500$$

(b) To verify this result with a calculator, compute the sum $a_1+a_2+a_3+\cdots+a_{10}$

$= 300000+301500+303000+304500+306000+307500+309000+310500$
$+312000+313500 = 3,067,500$

Example 19: A new car costs Rs.1200000. Assume that it depreciates 24% the first year, 20% the second year, 16% the third year and continues in the same manner for six years. If all the depreciations apply to the original cost, what is the value of the car in six years?

Solution: Since the depreciations 24%, 20%, 16%, ... form an arithmetic sequence with

$$a_1=24,\ d=-4 \text{ and } n=6$$

Calculating the sum of the depreciations over six years

$$S_n=\frac{n}{2}\{2a_1+(n-1)d\}$$

$$S_6=\frac{6}{2}\{48+5(-4)\}$$

$$=3(28)=84$$

Now the total depreciation in six years is 84% of 1200000

$$=\frac{84}{100}\times 1200000 = Rs.1008000$$

Thus the value of the car in six years = 1200000 − 1008000= Rs.192000.

Example 20: A display of cans in a grocery store consists of 24 cans in the bottom row, 21 cans in the next row and so on in an arithmetic sequence. The top row has 3 cans. Find the total number of cans in the display.

Solution: Since the display of cans are in arithmetic sequence with

$a_1 = 24$, $a_n = 3$ and $d = -3$ calculating the number of rows, we have

$$a_n = a_1+(n-1)d$$

$$3=24-3n+3$$

$$3n=24$$

$$n=8$$

Now the total number of cans is given by $S_n=\frac{n}{2}(a_1+a_n)$

$$S_8=\frac{8}{2}(24+3)$$

$$=4(27)$$

$$=108\ cans$$

EXERCISE 4.3

1. Find the indicated term and the sum of the indicated number of terms in case of each of the following arithmetic sequence:
   (i) 9, 7, 5, 3, … ; 20th term; 20 terms
   (ii) $3, \frac{8}{3}, \frac{7}{3}, 2, \ldots;$ 11^{th} term; 11 terms

2. Some of the components a_1, a_n, n, d and S_n are given. Find the ones that are missing:
   (i) $a_1 = 2, n = 17, d = 3$
   (ii) $a_1 = -40, S_{21} = 210$
   (iii) $a_1 = -7, d = 8, S_n = 225$
   (iv) $a_n = 4, S_{15} = 30$

3. Find the sum of all the numbers divisible by 5 from 25 through 350.

4. The sum of three numbers in an arithmetic sequence is 36 and the sum of their cubes is 6336. Find them. [Hint: suppose the numbers are $a - d$, a, $a + d$]

5. Find four numbers in arithmetic sequence, whose sum is 20 and the sum of whose squares is 120. [Hint: suppose the numbers are $a - 3d$, $a - d$, $a + d$, $a + 3d$]

6. $x_1, x_2, x_3, \ldots\ldots$ are in A.P. If $x_1 + x_7 + x_{10} = -6$ and $x_3 + x_8 + x_{12} = -11$, find $x_3 + x_8 + x_{22}$.

7. Find: 1+3–5+7+9–11+13+15–17+… up to 3n terms.

8. Show that the sum of the first n positive odd integers is n^2.

9. Find the sum of all multiples of 9 between 300 and 700.

10. The sum of Rs.1000 is distributed among four people so that each person after the first receives Rs. 20 less than the preceding person. How much does each person receive?

11. The distance which an object dropped from a cliff will fall 16ft the first second, 48 ft the next second, 80 ft the third second and so on. What is the total distance the object will fall in six seconds?

12. Afzal Khan saves Rs.1 the first day, Rs.2 the second, Rs.3 the third and Rs. N on the nth day for thirty days. How much does he save at the end of the thirtieth day?

13. A theater has 40 rows with 20 seats in the first row, 23 in the second row, 26 in the third row and so forth. How many seats are in the theater?

14. Insert enough arithmetic means between 1 and 50 so that the sum of the resulting series will be 459.

4.5 Geometric Sequence

4.5.1 In nature, certain phenomena can be described by geometric sequences. For example, archaeologists use the half–life of carbon 14 to estimate the age of ancient objects. Carbon 14 is a radioactive element that decays gradually, changing to nitrogen 14. The half–life (i.e. the time it takes for half of a given amount to decay) of carbon 14 is about 5600 years. Thus, one kg of carbon 14 will be reduced to $\frac{1}{2}$ kg in 5600 years, to $\frac{1}{4}$ kg in 11200 years, to $\frac{1}{8}$ kg in 16800 years and so on. Which is obviously a geometric sequence with r = $\frac{1}{2}$.

A geometric sequence (progression) is a sequence for which every term after the first is the product of the preceding term and a fixed number, called the common ratio of the sequence. We use the same notations as we use in A.P. with one exception that is instead of d, the common difference, we use r, the common ratio in geometric sequence.

Thus each of the following is a geometrical *sequence*.

3, 6, 12, 24, ...

$1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots\ldots\ldots$

$a, ar, ar^2, ar^3, \ldots\ldots\ldots, ar^{n-1}, \ldots\ldots\ldots$

The common ratio, and it is found by dividing any term by that which immediately precedes it.

In the first of the above examples the common ratio is 2; in the second it is $-\frac{1}{3}$; in the third it is r. A geometric sequence is recursively defined by equations of the form:

$$a_1 = a_1$$

and $a_{n+1} = ra_n$ where a_1 and r are real numbers, $a_1 \neq 0, r \neq 0$, and $n \in N$

4.5.2 The nth term of a Geometric Sequence

The nth term of a geometric sequence is given by: $a_n = a_1 r^{n-1}$

To find a formula for the nth term of a geometric sequence, we write down the first few terms using the recursive definition to observe the pattern:

$$\text{1st term} = a_1 = a_1 r^0 = a_1 r^{1-1}$$

$$\text{2nd term} = a_2 = a_1 r = a_1 r^{2-1}$$

$$\text{3rd term} = a_3 = a_2 r = a_1 r^2 = a_1 r^{3-1}$$

$$\text{4th term} = a_4 = a_3 r = a_1 r^3 = a_1 r^{4-1}$$

$$\vdots$$

$$\text{nth term} = a_n = a_1 r^{n-1}$$

Example 21: Find the first five terms and the tenth term of the geometric sequence having first term 3 and common ratio $-\frac{1}{2}$.

Solution: Here $a_1 = 3$, $r = -\frac{1}{2}$

Then the first five terms are

$$3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \frac{3}{16}$$

Substituting the values in the formula $a_n = a_1 r^{n-1}$

we have $a_{10} = (3)\left(-\frac{1}{2}\right)^{10-1}$ with n = 10

$$= 3\left(-\frac{1}{2}\right)^9 = -\frac{3}{512}$$

Example 22: Show that the sequence $\{a_n\} = 2^{-n}$ is geometric and find its' common ratio.

Solution: Since $a_n = 2^{-n}$

then $a_{n+1} = 2^{-(n+1)}$

and $r = \frac{a_{n+1}}{a_n} = \frac{2^{-(n+1)}}{2^{-n}} = \frac{1}{2}$

The ratio of successive terms is a nonzero number independent of n, thus $\{a_n\}$ is geometric sequence with $r = \frac{1}{2}$

Example 23: If the third term of a geometric sequence is 5 and the sixth term is − 40, find the eighth term.

Solution: Here $a_3 = 5$ and $a_6 = -40$ then we have $a_1 r^{3-1} = 5$ and $a_1 r^{6-1} = -40$

or $a_1 r^2 = 5$ (i)

and $a_1 r^5 = -40$ (ii)

Dividing the equation (ii) by the (i) we obtain

$$\frac{a_1 r^5}{a_1 r^2} = \frac{-40}{5}$$

or $\quad r^3 = -8 \;\; = (-2)^3 \Rightarrow r = -2$ and $\quad a_1 = \frac{5}{4}$ by (i)

Now $\quad a_8 = a_1 r^{8-1} \;\; = \left(\frac{5}{4}\right)(-2)^7 = -160$

4.6 Geometric Means (G.Ms)

4.6.1 when three numbers are in Geometrical Progression, the middle one is called the **geometric mean** between the other two.

Mathematically, it is derived in the following way:

Let a and b be the two numbers; G the geometric *mean then*

$\frac{b}{G} = \frac{G}{a}$, $\qquad$ since a, G, b are in G. P.,

$\therefore \; G^2 = ab;$

$G = \pm\sqrt{ab}.$

Example 24: Find the geometric mean of each of the following pairs of numbers.

(a) $\quad$ 9 and 16 $\qquad$ (b) $\quad \frac{-3}{10}$ and $-\frac{5}{6}$

Solution: $\quad$ (a) By the above definition

$$G = \sqrt{ab} \;\; = \sqrt{9 \times 16} \;\; = \sqrt{144} = 12$$, Since a,b >0 $\therefore$ G>0

(b) $\quad G = -\sqrt{ab}$ $\qquad$, Since a,b <0 $\therefore$ G<0

$$= -\sqrt{\left(\frac{-3}{10}\right)\left(\frac{-5}{6}\right)} \;\; = -\sqrt{\frac{1}{4}} = -\frac{1}{2}$$

4.6.2 To insert n Geometric Means between two numbers *a* and b

Since the terms between a and b of a geometric sequence are called the geometric means of a and b. Thus $G_1, G_2, \ldots, G_n$, are the n geometric means between a and b if $a, G_1, G_2, \ldots, G_n, b$ form a geometric sequence. Moreover it is a finite geometric sequence of n+2 terms with $a_1 = a$ and $a_{n+2} = b$.

Let r be its common ratio, then $a_{n+2} = b$ gives that $\; ar^{n+1} = b, \qquad \because a_n = ar^{n-1}$

$$\Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Hence $G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

$$\vdots$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Example 25: Insert two geometric means between 64 and 125.

Solution: Let G_1, G_2 be the two geometric means between 64 and 125 such that 64, G_1, G_2, 125 is a geometric sequence.

Thus $a_1 = 64, n = 4$ and $a_4 = 125$

Let r be the common ratio of the geometric sequence, then

$a_4 = 125$ gives $a_1 r^3 = 125$

$64r^3 = 125$ putting value of a_1

$$r^3 = \frac{125}{64} = \left(\frac{5}{4}\right)^3 \Rightarrow r = \frac{5}{4}$$

Hence $G_1 = a_1 r = (64)\left(\frac{5}{4}\right) = 80$ and $G_2 = a_1 r^2 = (64)\left(\frac{5}{4}\right)^2 = 100$

Example 26: Insert three geometric means between 2 and 32.

Solution: Let G_1, G_2, G_3 be the three geometric means between 2 and 32 such that

2, G_1, G_2, G_3, 32

is a geometric sequence.

We have $a_1 = 2, n = 5$ and $a_5 = 32$

let r be the common ratio, then

$$a_5 = 32$$

gives $a_1 r^4 = 32$

$$2r^4 = 32 \Rightarrow r^4 = 16 = (2)^4 \Rightarrow r = \pm 2$$

$\therefore$ we have two sets of geometric means given below: If r = 2

then $G_1 = a_1 r = (2)(2) = 4$

$G_2 = a_1r^2 = (2)(2)^2 = 8$

$G_3 = a_1r^3 = (2)(2)^3 = 16$

If $r = -2$ then

$G_1 = a_1r = (2)(-2) = -4$

$G_2 = a_1r^2 = (2)(-2)^2 = 8$

$G_3 = a_1r^3 = (2)(-2)^3 = -16$

Did You Know

It can be seen from Example 25 and 26 that if the number of required geometric means is even, a single set of geometric means is obtained, if the number of required geometric means is odd, two sets of geometric means are obtained.

EXERCISE 4.4

1. Write the first five terms of a geometric sequence given that:

   (i) $a_1 = 5; r = 3$ (ii) $a_1 = 8; r = -\frac{1}{2}$

   (iii) $a_1 = -\frac{9}{16}; r = -\frac{2}{3}$ (iv) $a_1 = \frac{x}{y}; r = -\frac{y}{x}$

2. Suppose that the third term of a geometric sequence is 27 and the fifth term is 243. Find the first term and common ratio of the sequence.
3. Find the seventh term of a geometric sequence that has 2 and $-\sqrt{2}$ for its second and third terms respectively.
4. How many terms are there in a geometric sequence in which the first and the last terms are 16 and $\frac{1}{64}$ respectively and $r = \frac{1}{2}$?
5. Find x so that $x+7, x-3, x-8$ form a three term geometric sequence in the given order. Also give the sequence.
6. If $a_{10} = \ell, a_{13} = m, a_{16} = n;$ show that $ln = m^2$
7. Show that the reciprocals of the terms of a geometric sequence also form a geometric sequence.
8. Find the geometric mean of the following:

   (i) 3.14 and 2.71 (ii) -6 and -216

   (iii) $x + y$ and $x - y$ (iv) $\sqrt{2}+3$ and $\sqrt{2}-3$

9. (i) Insert 5 geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$.

   (ii) Insert 6 geometric means between 14 and $-\frac{7}{64}$.

10. Find two numbers if the difference between them is 48 and their A.M. exceeds their G.M. by 18.

11. Prove that the product of n geometric means between a and b is equal to the nth power of the single geometric mean between them.

12. For what value of n, $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the geometric mean between a and b?

4.7 Geometric Series

4.7.1 Since with any geometric sequence we have an associated geometric series, which is the indicated sum of the terms of the geometric sequence.

Let $\{a_n\}$ is a geometric sequence, then $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + ... + a_n + ...$ is called a geometric series.

If r is the common ratio, then the above series can be written in the form

$$\sum_{i=1}^{\infty} a_1 r^{i-1} = a_1 + a_1 r + a_1 r^2 + ... a_1 r^{n-1} + ... \qquad (1)$$

known as the general form of the geometric series.

4.7.2 Sum of first n terms of a Geometric Series

Theorem: For a geometric sequence with first term a_1 and common ratio $r \neq 1$, the sum S_n of the first n terms is:

$$S_n = \frac{a_1(1-r^n)}{1-r} \qquad (2)$$

Proof: Let $S_n = a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-1}$

$S_n = a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-2} + a_1 r^{n-1}$ — S_n is the sum of the first n terms of the sequence.

$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^{n-1} + a_1 r^n$ — Multiply both sides of the equation by r.

$S_n - rS_n = a_1 - a_1 r^n$ — Subtract the second equation from the first equation

$S_n(1-r) = a_1(1-r^n)$ — Factor out S_n on the left and a_1 on the right.

$S_n = a_1 \dfrac{(1-r^n)}{1-r}$ — Solve for S_n by dividing both sides by $1-r$ (assuming that $r \neq 1$).

which is the required sum of the first n terms of a geometric sequence.

Since $S_n = \dfrac{a_1(1-r^n)}{1-r}$

$$= \frac{a_1 - a_1 r^n}{1-r} = \frac{a_1 - (a_1 r^{n-1})r}{1-r}$$

$$= \frac{a_1 - a_n r}{1-r}, a_n = a_1 r^{n-1} \text{ is the last term}$$

$$= \frac{a_1 - a_n r}{1-r}, r \neq 1 \qquad (3)$$

is the alternative form of the result given in (2)

Remember

It is better to use the forms

(i) $S_n = \dfrac{a_1(1-r^n)}{1-r}$ and

$S_n = \dfrac{a_1 - a_n r}{1-r}$, whenever $|r| < 1$

(ii) If $|r| > 1$ then the following forms are used $S_n = \dfrac{a_1(r^n - 1)}{r-1}$

and $S_n = \dfrac{a_n r - a_1}{r-1}$

because the numerators and denominators are positive.

(iii) If r=1, we have the trivial geometric series:

$S_n = a_1 + a_1 + ... + a_1 = na_1$

Example 27: Approximate the sum for the given values of n.

(a) 1 +1/2 +1/4 + …+$(1/2)^{n-1}$; n=5, 10, and 20

(b) 3 − 6 + 12 − 24 + 48 - …. +$3(-2)^{n-1}$;

n = 3, 8, and 13

Solution: (a) This geometric series has

$a_1 = 1$ and $r = ½ = 0.5$.

$$S_5 = \frac{1(1-0.5^5)}{1-0.5} = 1.9375$$

$$S_{10} = \frac{1(1-0.5^{10})}{1-0.5} = 1.998047$$

$$S_{20} = \frac{1(1-0.5^{20})}{1-0.5} = 1.999998$$

(b) This geometric series has $a_1 = 3$ and $r = \dfrac{-6}{3} = -2$

$$S_3 = \frac{3(1-(-2)^3}{1-(-2)} = 9$$

$$S_8 = \frac{3(1-(-2)^8}{1-(-2)} = -255$$

$$S_{13} = \frac{3(1-(-2)^{13}}{1-(-2)} = 8193$$

Example 28: Sum the series $\frac{2}{3}, -1, \frac{3}{2}, \ldots\ldots\ldots to\ 7$ terms.

Solution: The common ratio $= -\frac{3}{2}$; hence by formula $\frac{a_1(1-r^n)}{1-r}$

$$\text{The sum } = \frac{\frac{2}{3}\left\{1-\left(-\frac{3}{2}\right)^7\right\}}{1+\frac{3}{2}} = \frac{\frac{2}{3}\left\{1-\left(-\frac{3}{2}\right)^7\right\}}{1+\frac{3}{2}}$$

$$= \frac{\frac{2}{3}\left\{1+\frac{2187}{128}\right\}}{\frac{5}{2}} = \frac{2}{3}\times\frac{2315}{128}\times\frac{2}{5} = \frac{463}{95}.$$

Example 29: Compute: 2+6+18+54+162+486

Solution: In this case $a_1 = 2, r = \frac{6}{2} = 3 > 1, n = 6$

Substituting the values in

$$S_n = \frac{a_1(r^n-1)}{r-1}$$

$$S_6 = \frac{2(3^6-1)}{3-1} = 729 - 1 = 728$$

Example 30: Given that $a_1 = \frac{3}{4}, a_n = 48$ and $S_n = 32\frac{1}{4}$, find r and n

Solution: Since $a_1 = \frac{3}{4}, a_n = 48$

Then $a_1 r^{n-1} = 48$

and $\frac{3}{4} r^{n-1} = 48$

$\Rightarrow$ $r^{n-1} = 64$ (i)

Also, we have $S_n = \frac{a_1 - ra_n}{1-r}$

$$\frac{129}{4} = \frac{\frac{3}{4} - 48r}{1-r}$$

$$129 - 129r = 3 - 192r$$

$$63r = -126$$

$$r = -2$$

From (i) we have

$$(-2)^{n-1} = 64 \Rightarrow (-2)^{n-1} = (-2)^6$$

$$n-1 = 6 \Rightarrow \quad n = 7$$

Example 31: Suppose that the third term of a geometric sequence is 27 and the fifth term is 243. Find a_1, r and S_5.

Solution: Since $a_3 = 27$ and $a_5 = 243$

Then we have $a_1r^2 = 27$ (i) $\quad a_1r^4 = 243$ (ii),

$$\therefore \quad a_n = a_1r^{n-1}$$

Dividing (ii) by (i) we obtain

$$\frac{a_1r^4}{a_1r^2} = \frac{243}{27}$$

$$r^2 = 9 \Rightarrow r = \pm 3$$

We obtain two different solutions since there are two values of r.

$$r = 3$$

$$a_1r^2 = 27$$

$$a_1(3)^2 = 27$$

$$a_1.9 = 27$$

$$a_1 = 3$$

The first sequence is

3,9,27,81,243,...

$$S_5 = \frac{ra_5 - a_1}{r-1}$$

$$= \frac{(3)(243) - 3}{3-1}$$

$$= \frac{729-3}{2} = 363$$

$$r = -3$$

$$a_1r^2 = 27$$

$$a_1(-3)^2 = 27$$

$$a_1.9 = 27$$

$$a_1 = 3$$

The second sequence is

3,–9,27,–81,243,...

$$S_5 = \frac{ra_5 - a_1}{r-1}$$

$$S_5 = \frac{(-3)(243) - 3}{-3-1}$$

$$= \frac{-732}{-4} = 183$$

Example 32: Find the sum $\sum_{i=1}^{10} 6\cdot 2^i$

Solution: $\sum_{i=1}^{10} 6\cdot 2^i = 6\cdot 2 + 6\cdot 2^2 + 6\cdot 2^3 + \cdots + 6\cdot 2^{10}$

Do you see that each term after the first is obtained by multiplying the preceding term by 2? To find the sum of the 10 terms (n=10), we need to know the first term, a_1, and the common ratio, r. The first term is 6. 2 or 12: a_1= 12. The common ratio is 2.

$S_n = \dfrac{a_1(r^n-1)}{r-1}$	Use the formula for the sum of the first n terms of a geometric sequence.
$S_{10} = \dfrac{12\,(2^{10}-1)}{2-1}$	a_1 (the first term) = 12, r = 2, and n = 10 because are adding ten terms
= 12,276	Use a calculator

Thus, $\sum_{i=1}^{10} 6\cdot 2^i = 12,276$

4.7.3 Sum of infinite Geometric series

Our discussion of series has so far been restricted to those associated with finite sequences. The series associated with the infinite sequence:

$$a_1, a_1r, a_1r^2, ..., a_1r^{n-1}, ...$$

is denoted by:

$$a_1 + a_1r + a_1r^2 + ..., a_1r^{n-1} + ... = \sum_{i=1}^{\infty} a_1r^{i-1}$$

and is called an infinite series. Important questions arise over here are, what do we mean by the "sum" of an infinite number of terms, and under what circumstances does such a "sum" exist? The answers to these questions depend upon the concept of "limit" which is studied in a course in the calculus. However, for some particular infinite series we can give an intuitive idea of the concept of "sum".

Consider the formula for the sum of the first n terms in a geometric sequence, we have already proved that:

$$S_n = \begin{cases} a_1 + a_1 + a_1 + \ldots + a_1 = na_1, r = 1 \\ a_1 - a_1 + a_1 - \ldots + (-1)^{n-1} a_1, r = -1 \\ \dfrac{a_1(r^n - 1)}{r - 1}, |r| > 1 \\ \dfrac{a_1(1 - r^n)}{1 - r}, |r| < 1 \end{cases} \qquad (4)$$

(i) Since $S_n = na_1$, when r=1

As n increases, the sum of the infinite geometric series increases without limit. Symbolically it is written as:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} na_1 = \infty$$

Thus the infinite geometric series in this case does not have a finite sum.

(ii) Here $S_n = a_1 - a_1 + a_1 - \ldots + (-1)^{n-1} a_1$ when $r = -1$

The sum of the first n terms is $a_1$or 0 according as n is odd or even; therefore the sum oscillates between the values 0 and a_1.

(iii) $$S_n = \frac{a_1(r^n - 1)}{r - 1}, |r| > 1 = \frac{a_1 r^n}{1 - r} - \frac{a}{r - 1}, |r| > 1$$

Since $|r| > 1$, then the absolute value of each term is greater than the absolute value of the preceding term. Therefore such an infinite series cannot have a finite "sum".

Mathematically, it is shown that:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{a_1 r^n}{1 - r} - \frac{a_1}{1 - r} \right) = \lim_{n\to\infty} \left(\frac{a_1 r^n}{1 - r} \right) - \frac{a_1}{1 - r} = \infty - \frac{a_1}{1 - r} = \infty$$

(iv) $$S_n = \frac{a_1(1 - r^n)}{1 - r}, |r| < 1$$

$$= \frac{a_1}{1 - r} - \frac{a_1 r^n}{1 - r}, |r| < 1$$

This is the case which provides us a quite different situation and we have some useful result.

Since $|r| < 1$, then r^n approaches zero as n increases with out bound, that is, we can make r^n or $\dfrac{ar^n}{1 - r}$ as close as we wish to 0 by taking n sufficiently large. It follows that S_n approaches $\frac{a_1}{1-r}$ as n increases without a bound and we write

$$S_\infty = \frac{a_1}{1-r}$$

Mathematically, it can be shown as $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{a_1}{1-r} - \frac{a_1 r^n}{1-r}\right)$

$$= \frac{a_1}{1-r} - \frac{a_1}{1-r}\lim_{n\to\infty} r^n$$

$$= \frac{a_1}{1-r} - \frac{a_1}{1-r}(0)$$

$$S_\infty = \frac{a_1}{1-r}$$

This gives us the following:

Theorem: If $|r|<1$, then the infinite geometric series:
$a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + \cdots$ has the sum: $\frac{a_1}{1-r}$

Example 33: Find the sum of the infinite geometric series:
$\frac{3}{8} - \frac{3}{16} + \frac{3}{32} - \frac{3}{64} + \cdots$

Solution: Before finding the sum, we must find the common ratio.

$$r = \frac{a_2}{a_1} = -\frac{3/16}{3/8} = -\frac{3}{16}\cdot\frac{8}{3} = -\frac{1}{2}$$

Because r = −1/2, the condition that | r | < 1 is met. Thus, the infinite geometric series has a sum

$$S = \frac{a_1}{1-r}$$

This is the formula for the sum of an infinite geometric series. Here $a_1 = 3/8$ and r = −1/2

$$= \frac{3/8}{1-(-1/2)} = \frac{3/8}{3/2} = \frac{3}{8}\cdot\frac{2}{3} = \frac{1}{4}$$

Thus, the sum of $\frac{3}{8} - \frac{3}{16} + \frac{3}{32} + \ldots\ldots\ldots$ is $\frac{1}{4}$. Put in an informal way, as we continue to add more and more terms, the sum is approximately $\frac{1}{4}$.

Example 34: The sum of an infinite number of terms in G. P. is 15, and the sum of their squares is 45. Find the series.

Solution: Let a denote the first term, r the common ratio; then the sum of the term is $\frac{a}{1-r}$; and the sum of their square is $\frac{a^2}{1-r^2}$.

Hence $\frac{a}{1-r}=15$.......................(1)

$\frac{a^2}{1-r^2}=45$.......................(2)

Dividing (2) by (1) $\frac{a}{1+r}=3$.......................(3)

And from (1) and (3) $\frac{1+r}{1-r}=5;\quad \Rightarrow \quad r=\frac{2}{3}$, and therefore $a=5$.

Thus the series is $5, \frac{10}{3}, \frac{20}{9}, \ldots\ldots$

Example 35: Find the sum of the infinite geometric sequence: $1, \frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{3^n}, \ldots$

Solution: Here $a_1=1, r=\frac{1}{3}$ and $|r|=\frac{1}{3}<1$

Thus the sum exists and is given by the formula:

$$S_\infty = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

4.7.4 Conversion of recurring Decimals into an equivalent fraction

Recurring decimals furnish a good illustration of infinite Geometrical Progressions.

Example 36: Convert $2.3\overline{4}$ to a common fraction.

Solution: Since $2.3\overline{4} = 2.3 + 0.0\overline{4}$

$$=\frac{23}{10}+0.04444\ldots\ldots$$

$$=\frac{23}{10}+0.04+0.004+0.0004+\ldots\ldots$$

$$=\frac{23}{10}+\left(\frac{a_1}{1-r}\right), a_1=0.04, |r|=0.1<1$$

$$=\frac{23}{10}+\left(\frac{0.04}{1-0.1}\right)=\frac{23}{10}+\frac{4}{90}=\frac{211}{90}$$

Example 37: Convert $0.\overline{21}$ to a common fraction

Solution: Since $0.\overline{21} = 0.212121\ldots = 0.21 + 0.0021 + 0.000021 + \ldots$

$$= 0.21 + (0.01)(0.21) + (0.01)^2(0.21) + \ldots$$

Which is an infinite geometric series with $a_1 = 0.21$, $r = 0.01$,

and $|r| = 0.01 < 1$, so the sum exists and is given by $S = \dfrac{a_1}{1-r} = \dfrac{0.21}{1-0.01} = \dfrac{7}{33}$

Thus $0.\overline{21} = \dfrac{7}{33}$

4.7.5 Real life problems involving Geometric series

Example 38: Computing a lifetime salary

A union contract specifies that each worker will receive a 5 % pay increase each year for the next 30 years. One worker is paid Rs. 20,000 the first year. What is this person's total lifetime salary over a 30-years period?

Solution: The salary for the first year is 20,000. With a 5% raise, the second-year salary is computed as follows:

Salary for year 2 = 20,000 + 20,000(0.05) = 20,000(1 + 0.05) = 20,000(1.05).

Each year, the salary is 1.05 times what it was in the previous year. Thus, the salary for year 3 is 1.05 times 20,000(1.05), or $20,000(1.05)^2$. Thus

Yearly Salaries					
Year 1	**Year 2**	**Year 3**	**Year 4**	**Year 5**	...
20,000	20,000(1.05)	$20,000(1.05)^2$	$20,000(1.05)^3$	$20,000(1.05)^4$	...

The numbers in the bottom row form a geometric sequence with $a_1 = 20,000$ and r = 1+5 %=1+.05=1.05. To find the total salary over 30 years, we use the formula for the sum of the first n terms of a geometric sequence, with n = 30.

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{20,000[1-(1.05)^{30}]}{1-1.05} = \frac{20,000[1-(1.05)^{30}]}{-0.05} \approx 1,328,777$$

(Use a calculator)

The total salary over the 30-years period is approximately Rs. 1,328,777.

Example 39: The tip of a pendulum moves back and forth so that it sweeps out an arc 12 inches in length and on each succeeding pass, the length of the arc traveled, is $\frac{7}{8}$ of the length of the preceding pass. What is the total distance traveled by the tip of the pendulum?

Solution: Since the pendulum eventually comes to rest due to friction. We have the following geometric infinite sequence.

$$12, \left(\frac{7}{8}\right)(12), \left(\frac{7}{8}\right)^2(12), \left(\frac{7}{8}\right)^3(12), \ldots$$

and the total distance traveled $S = 12 + \left(\frac{7}{8}\right)(12) + \left(\frac{7}{8}\right)^2(12) + \left(\frac{7}{8}\right)^3(12) + \ldots$

which is an infinite geometric series with

$a_1 = 12, r = \frac{7}{8}$ and $|r| < 1$, so the sum exists.

Thus the total distance traveled $= \frac{a_1}{1-r}$

$$= \frac{12}{1-\frac{7}{8}}$$

$= 96$ inches

Did You Know

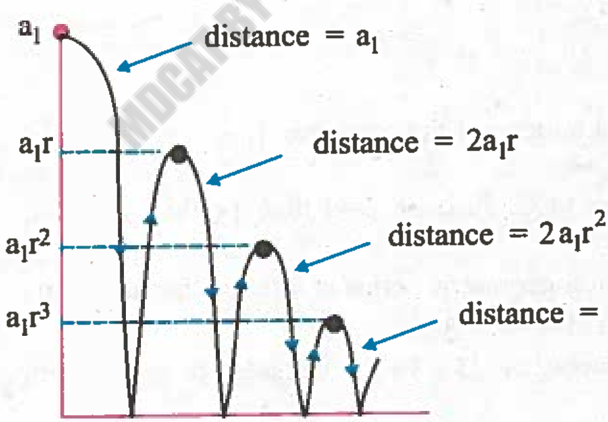
The symbol ∞ (infinity) is merely a notational device and does not represent a real number. Loosely, it is the concept of a value beyond any finite value.

Example 40:
A ball is dropped from x feet above a flat surface. Each time the ball hits the ground after falling a distance h, it rebounds a distance rh where $r < 1$. Compute the total distance the ball travels.

Solution: The path and the distance the ball travels is shown on the sketch of figure. The total distance s is computed by the geometric series

$s = a_1 + 2a_1r + 2a_1r^2 + 2a_1r^3 + \ldots$ (I)

The common ratio is $\frac{2a_1r}{1-r}$ (II)



Adding the first term of (I) with (II) we form the total distance as

$$s = a_1 + \frac{2a_1 r}{1-r} = a_1\left(\frac{1+r}{1-r}\right)$$

For example, if $a_1 = 6$ ft and $r = 2/3$, the total distance the ball travels is

$$s = 6 \times \frac{1+2/3}{1+2/3} = 30 \text{ ft}$$

EXERCISE 4.5

1. Compute the sum:

   (i) $3+6+12+\ldots+3.2^9$ (ii) $8+4+2+1+\ldots+\frac{1}{16}$

   (iii) $2^4+2^5+2^6+\ldots+2^{10}$ (iv) $\frac{8}{5},-1,\frac{5}{8},\ldots$

   (v) $2,\frac{2}{\sqrt{2}},1,\frac{1}{\sqrt{2}},\frac{1}{2},\ldots$ (vi) $-\frac{1}{3},\frac{1}{2},-\frac{3}{4},\ldots\ldots\ldots\ldots$ to 7 terms.

2. Some of the components a_1, a_n, n, r and S_n of a geometric sequence are given. Find the ones that are missing.

   (i) $a_1 = 1, r = -2, a_n = 64$ (ii) $r = \frac{1}{2}, a_9 = 1$

   (iii) $r = -2, S_n = -63, a_n = -96$

3. Find the first five terms and the sum of an infinite geometric sequence having $a_2 = 2$ and $a_3 = 1$

4. Find the value of: (i) $0.\bar{8}$ (ii) $1.\overline{63}$ (iii) $2.1\bar{5}$ (iv) $0.\overline{123}$

5. Find r such that: $S_{10} = 244S_5$.

6. Prove that: $S_n(S_{3n} - S_{2n}) = (S_n - S_{2n})^2$

7. Find the sum S_n of the first n terms of the sequence $\left\{\left(\frac{1}{2}\right)^n\right\}$.

8. The sum of three numbers in G. P. is 38, and their product is 1728; find them.

9. The sum of first 6 terms of a geometric series is 9 times the sum of its first three terms. Find the common ratio.

10. How many terms of the series: $1+\sqrt{3}+3+\ldots.$ be added to get the sum $40+13\sqrt{3}.$

11. If p^{th}, q^{th}, r^{th} terms of a G. P. be a,b,c respectively,prove that $a^{q-r}b^{r-p}c^{p-q}=1$.

12. Find an infinite geometric series whose sum is 6 and such that each term is four times the sum of all the terms that follow it.

13. If $y=\frac{x}{3}+\frac{x^2}{3^2}+\frac{x^3}{3^3}+...,$ where $0<x<3$, then show that $x=\frac{3y}{1+y}$

14. A ball rebounds to half the height from which it is dropped. If it is dropped from 10 ft, how far does it travel from the moment it is dropped until the moment of its eighth bounce?

15. A man wishes to save money by setting aside Rs.1 the first day, Rs.2 the second day, Rs.4 the third day and so on, doubling the amount each day. If this continued, how much must be set aside on the 15th day? What is the total amount saved at the end of 30 days?

16. The number of bacteria in a culture increased geometrically from 64000 to 729000 in 6 days. Find the daily rate of increase if the rate is assumed to be constant.

4.8 Harmonic Sequence

4.8.1 A harmonic sequence is a sequence whose reciprocals form an arithmetic sequence.

The sequence: $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}$ (1)

is not an arithmetic sequence. However the reciprocals of these numbers, namely: 2, 4, 6, 8,10 do form an arithmetic sequence. Thus the sequence (1) is an example of a harmonic sequence. A harmonic sequence is also called a harmonic progression (H.P).

Example 41: Three numbers *a,b,c* are in H.P. when $\frac{a}{c}=\frac{a-b}{b-c}$

Solution: Given $\frac{a}{c}=\frac{a-b}{b-c}$ then $a(b-c)=c(a-b)$

or $ab-ac=ca-bc$ Dividing by (abc), we obtain:

$$\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a}$$

Thus $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. and hence *a*, b, c are in H.P.

4.8.2 Finding nth Term of a Harmonic Sequence

The typical form of a harmonic sequence is

$$\frac{1}{a_1}, \frac{1}{a_1+d}, \frac{1}{a_1+2d}, \ldots, \frac{1}{a_1+(n-1)d}, \ldots$$

The general term or the nth term of this H.P. is

$$\frac{1}{a_1+(n-1)d}$$

Remember

Many properties of harmonic progression can be obtained from the corresponding arithmetic progression. However, there is no elementary formula for the sum of a harmonic sequence.

whose reciprocal $a_1+(n-1)d$ is the nth term of the A.P.

Example 42: Find the twelveth term of the harmonic progression: 6, 4, 3,…

Solution: The 12^{th} term of the corresponding A.P.

$$\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \ldots \quad \text{twelvth}$$

with $a_1 = \frac{1}{6}$, $d = \frac{1}{12}$, $n = 12$

is $a_{12} = \frac{1}{6} + (12-1)\left(\frac{1}{12}\right)$ $\qquad \therefore \quad a_n = a_1 + (n-1)d$

$$= \frac{13}{12}$$

Thus the 12^{th} term of the given H.P is $\frac{12}{13}$.

4.9 Harmonic Means (H.Ms)

4.9.1 (i) A number H is said to be the Harmonic Mean (H.M) between two number a and b $(a \neq 0, b \neq 0)$ if a, H, b are in H.P.

Then $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P. and $\frac{1}{H} = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$, i.e. $\frac{1}{H}$ is the A.M between $\frac{1}{a}$ and $\frac{1}{b}$.

$$\frac{1}{H} = \frac{a+b}{2ab}$$

$$\therefore \quad H = \frac{2ab}{a+b} \quad \text{is the H.M. between } a \text{ and b}$$

(ii) The numbers $H_1, H_2, \ldots, H_n$ are said to be the n Harmonic Means (H.Ms) between two number a and b $(a \neq 0, b \neq 0)$ if

$$a, H_1, H_2, H_3, ..., H_n, b \text{ are in H.P.}$$

Then obviously: $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, ... \frac{1}{H_n}, \frac{1}{b}$ are in A.P with n+2 terms.

$\therefore \quad \frac{1}{a} + (n+2-1)d = \frac{1}{b}$, utilizing $a_n = a_1 + (n-1)d$

$$\Rightarrow d = \frac{a-b}{ab(n+1)}$$

Thus $\frac{1}{H_1} = \frac{1}{a} + \frac{a-b}{ab(n+1)}$ or $H_1 = \frac{ab(n+1)}{nb+a}$

$\frac{1}{H_2} = \frac{1}{a} + 2\frac{a-b}{ab(n+1)}$ or $H_2 = \frac{ab(n+1)}{(n-1)b+2a}$

$\frac{1}{H_3} = \frac{1}{a} + 3\frac{a-b}{ab(n+1)}$ or $H_3 = \frac{ab(n+1)}{(n-2)b+3a}$

.
.
.

$\frac{1}{H_n} = \frac{1}{a} + n\frac{a-b}{ab(n+1)}$ or $H_n = \frac{ab(n+1)}{b+na}$

by using $\frac{1}{H_i} = a_1 + id, \quad i = 1, 2, 3, ..., n$. Hence $H_1, H_2, H_3, ..., H_n$, are the n H.Ms between a and b.

Example 43: Find the harmonic mean of 24 and 16

Solution: $H = \frac{2ab}{a+b}$, where $a = 24$, b = 16

Then $H = \frac{2(24)(16)}{24+16} = \frac{2 \times 24 \times 16}{40} = \frac{96}{5}$

Example 44: Insert four harmonic means between $-\frac{1}{2}$ and $\frac{1}{13}$.

Solution: Let H_1, H_2, H_3 and H_4 be the required H.Ms, then

$-\frac{1}{2}, H_1, H_2, H_3, H_4, \frac{1}{13}$ are in H.P

$\therefore \quad -2, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, 13$ are in A.P with $a_1 = -2$

$a_6 = 13$

$a_1 + 5d = 13$

$-2 + 5d = 13$

$d = 3$

Now $\frac{1}{H_1} = -2 + 3 = 1 \Rightarrow H_1 = 1$

$\frac{1}{H_2} = 1 + 3 = 4 \Rightarrow H_2 = \frac{1}{4}$

$\frac{1}{H_3} = 4 + 3 = 7 \Rightarrow H_3 = \frac{1}{7}$

$\frac{1}{H_4} = 7 + 3 = 10 \Rightarrow H_4 = \frac{1}{10}$

Hence $1, \frac{1}{4}, \frac{1}{7}, \frac{1}{10}$ are the required 4 H.Ms. between $-\frac{1}{2}$ *and* $\frac{1}{13}$

Did You Know

Harmonic sequences are called harmonic because of their use in the study of musical chords and their relationship, that is, harmony.

Example 45: **Find a relation among Arithmetic, Geometric And Harmonic Means.**

Solution: Let $a \neq 0, b \neq 0$ be any two positive numbers,

then $A = \frac{a+b}{2}$

$H = \frac{2ab}{a+b}$ and $G = \sqrt{ab}$

(i) $A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab = \left(\sqrt{ab}\right)^2 = G^2 \Rightarrow A, G, H$ are in G.P

(ii) A > G if $\frac{a+b}{2} > \sqrt{ab}$

$a + b > 2\sqrt{ab}$

$a + b - 2\sqrt{ab} > 0,$

$(\sqrt{a} - \sqrt{b})^2 > 0$, which is always true

$\therefore \quad A > G \qquad\qquad (1)$

$G > H$ if $\sqrt{ab} > \dfrac{2ab}{a+b}$

$a + b > 2\sqrt{ab}$

$(\sqrt{a} - \sqrt{b})^2 > 0$, which is always true

$\therefore \quad G > H \qquad\qquad (2)$

(1) and (2) $\Rightarrow A > G > H$

EXERCISE 4.6

1. Find the indicated term of each of the following harmonic progressions:

   (i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$; 9^{th} term (ii) $6, 2, \frac{6}{5}, \ldots$; 20^{th} term

   (iii) $5\frac{2}{3}, 3\frac{2}{5}, 2\frac{3}{7}, \ldots$; 8^{th} term

2. Find five more terms of the H.P. $\frac{1}{3}, 1, -1, \ldots$

3. The second term of an H.P is $\frac{1}{2}$ and the fifth term is $-\frac{1}{4}$. Find the 12^{th} term.

4. Find the arithmetic, harmonic and geometric means of each of the following. Also verify that $A \times H = G^2$.

   (i) 3.14 and 2.71 (ii) -6 and -216 (iii) $x + y$ and $x - y$

5. For what value of n will $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ be the harmonic mean between a and b?

6. Insert two harmonic means between 12 and 48.

7. Insert four harmonic means between $\frac{7}{3}$ and $\frac{7}{11}$.

8. Prove that the square of the geometric mean of two numbers equals the product of the arithmetic mean and the harmonic mean of the two numbers.

9. The arithmetic mean of two numbers is 8, and the harmonic mean is 6. What are the numbers?

10. The harmonic mean of two numbers is $4\frac{4}{5}$ and the geometric mean is 6. What are the numbers?

Review Exercise 4

1. Choose the correct option.

(i) The sum to 200 terms of the series 1+ 4+ 6+ 5+ 11+ 6+ …….is
(a) 20,300 (b) 29,800
(c) 30,200 (d) None of these

(ii) If the sum of the series 2+ 5+ 8+ 11……is 60100, then the number of the terms is
(a) 100 (b) 200 (c) 150 (d) 250

(iii) If a, b, c are in G.P., then
(a) a^2, b^2, c^2 are in G.P. (b) $a^2(b+c), c^2(a+b), b^2(a+c)$ are in G.P.
(c) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in G.P. (d) None of these

(iv) If the nth term of an A.P is 4n+ 1, then the common difference is :
(a) 3 (b) 4 (c) 5 (d) 6

(v) Which of the following is not a G.P.?
(a) 2, 4, 8, 16, …….. (b) 5, 25, 125, 625, ……
(c) 1.5, 3.0, 6.0, 12.0 ….. (d) 8, 16, 24, 32, ………

(vi) There are four arithmetic means between 2 and −18. The means are
(a) −4, −7, −10, −13 (b) 1, −4, −7, −10
(c) −2, −5, −9, −13 (d) −2, −6, −10, −14

(vii) If A, G and H are A.M, G.M and H.M. of any two positive numbers, then find the relation between A, G and H.
(a) $A^2 = GH$ (b) $G^2 = AH$ (c) $H^2 = AG$ (d) $G^2 = A^2H$

(viii) Find the number of terms to be added in the series 27, 9, 3, …… so that the sum is 1093/27
(a) 6 (b) 7 (c) 8 (d) 9

(ix) Find the value of p (p>0) if $\frac{15}{4}+p, \frac{5}{2}+2p$ and 2+p are the three consecutive terms of a geometric progression
(a) $\frac{3}{4}$ (b) $\frac{1}{4}$ (c) $\frac{5}{3}$ (d) $\frac{1}{2}$

(x) The 10^{th} term of harmonic progression 1/5, 4/19, 2/9, 4/17,…is
(a) 11/4 (b) 13/4 (c) 4/13 (d) 4/11

(xi) Find the sum of 3 geometric means between 1/3 and 1/48 (r > 0).
(a) 1/4 (b) 5/24 (c) 7/24 (d) 1/3

2. If the first term and common difference in an A.P are 8 and −1 respectively, then find:

   (i) General term (ii) The Progression (iii) The 10th term and
   (iv) The expression for sum to n terms and hence sum to 10 terms.

3. If the sum of the n terms of the series 54, 51, 48, ... is 513, then find the value of n.

4. If the sum of n terms of an A.P. is $2n+3n^2$, generate the progression and find the nth term

5. Find the sum of all natural numbers between 250 and 1000 which are exactly divisible by 3.

6. Find the sum of the series 1, 2/5, 4/25, 8/125, ..., ∞

7. If a, b, c are in A.P. and x, y, z are in G.P, show that $x^b y^c z^a = x^c y^a z^b$

8. Find the arithmetic mean between $10\frac{1}{2}$ and $25\frac{1}{2}$.

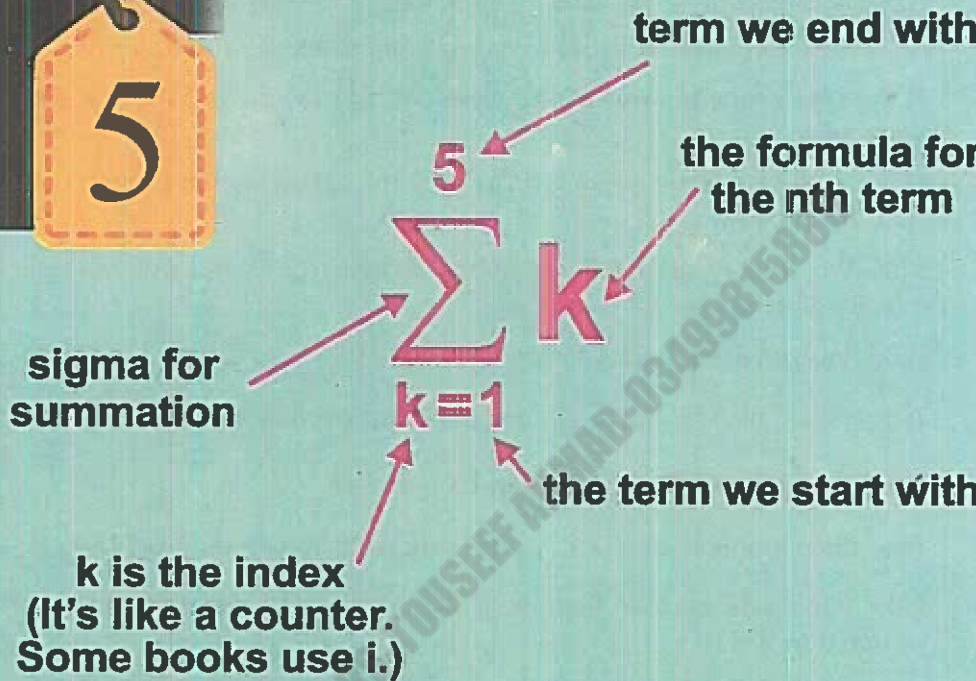
9. Find three numbers of a G.P. whose sum is 26 and product is 216.

10. How many odd integers beginning with 15 must be taken for their sum to be equal to 975?

11. A gas-filled balloon has risen 100 feet. In each succeeding minute, the balloon rises only 50% as far as it rose in the previous minute. How far will it rise in 5 minutes?

UNIT 5

MISCELLANEOUS SERIES



STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Recognize sigma (Σ) notation.
- Find sum of
  - the first n natural numbers (Σn),
  - the squares of the first n natural numbers (Σn^2),
  - the cubes of the first n natural numbers (Σn^3).
- Define arithmetico-geometric series.
- Find sum to n terms of the arithmetico-geometric series.
- Define method of differences. Use this method to find the sum of n terms of the series whose differences of the consecutive terms are either in arithmetic or in geometric sequence.
- Use partial fractions to find the sum to n terms and to infinity the series of the type $\dfrac{1}{a(a+d)}+\dfrac{1}{(a+d)(a+2d)}+\ldots$

5.1 Introduction

In the previous chapter, we computed the sums of arithmetic and geometric sequences. In this chapter, we discuss a few more techniques for computing sums of some other sequences. Since we are already familiar with the standard notation, called the sigma notation (Σ) and its rules. However here we properly define it with a few examples of summation notation.

5.1.1 Sigma Notation

The letter "Σ" of the Greek alphabet (pronounced as sigma) is used to denote the sum of a given series. The letter Σ is placed before the rth term, say, a_r. We, thus write Σa_r to denote the sum of terms of the type a_r. If we want to sum up terms a_r for values of r corresponding to $r = 1, 2, 3, ...n$, we denote the sum by

$$\sum_{r=1}^{r=n} a_r \text{ or by } \sum_{1}^{n} a_r$$

Example 1: Find the following sum.

(i) $\sum_{k=1}^{4} k^2(k-3)$ (ii) $\sum_{k=0}^{3} \frac{2^k}{(k+1)}$ (iii) $\sum_{k=2}^{6} (-1)^k \sqrt{k}$

Solution: (i) $\sum_{k=1}^{4} k^2(k-3) = 1^2(1-3) + 2^2(2-3) + 3^2(3-3) + 4^2(4-3)$

$= (-2) + (-4) + 0 + 16 = 10$

(ii) $\sum_{k=0}^{3} \frac{2^k}{(k+1)} = \frac{2^0}{(0+1)} + \frac{2^1}{(1+1)} + \frac{2^2}{(2+1)} + \frac{2^3}{(3+1)}$

$= 1 + 1 + \frac{4}{3} + 2 = \frac{16}{3}$

(iii) $\sum_{k=2}^{6} (-1)^k \sqrt{k}$

$= (-1)^2\sqrt{2} + (-1)^3\sqrt{3} + (-1)^4\sqrt{4} + (-1)^5\sqrt{5} + (-1)^6\sqrt{6}$

$= \sqrt{2} - \sqrt{3} + 2 - \sqrt{5} + \sqrt{6}$

Example 2: Simplify $\sum_{j=2}^{10} \frac{1}{j} - \sum_{j=1}^{8} \frac{1}{j+2}$

Solution: It can be seen that most terms are common to both sums and will cancel. In the second sum, let $k = j+2$ and in the first sum, let $k = j$, then we have

$$\sum_{j=2}^{10} \frac{1}{j} - \sum_{j=1}^{8} \frac{1}{j+2} = \sum_{k=2}^{10} \frac{1}{k} - \sum_{k=3}^{10} \frac{1}{k}$$

$$= \frac{1}{2} + \sum_{k=3}^{10} \frac{1}{k} - \sum_{k=3}^{10} \frac{1}{k} = \frac{1}{2}$$

This example illustrates how changing the index can simplify expressions involving several sums.

5.1.2 Evaluation of sum of the first n

i. Natural numbers
ii. Squares of natural numbers
iii. Cubes of natural numbers

Before evaluation of the above mentioned sums, here we discuss a general principle that will allow us to compute a wide variety of sums.

Suppose $b_1, b_2, \ldots, b_{n+1}$ is a sequence

and $a_j = b_{j+1} - b_j$

then
$$\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} (b_{j+1} - b_j)$$
$$= (b_2 - b_1) + (b_3 - b_2) + \ldots + (b_{n+1} - b_n)$$
$$= -b_1 + (b_2 - b_2) + (b_3 - b_3) + \ldots + (b_n - b_n) + b_{n+1}$$
$$= b_{n+1} - b_1$$

Thus if $a_j = b_{j+1} - b_j$

then $\sum_{j=1}^{n} a_j = b_{n+1} - b_1$

This statement seems very simple, yet in practice it can be very powerful.

Suppose we want to compute $\sum_{j=1}^{n} a_j$. If we can find a sequence $b_1, b_2, \ldots$ such that $b_{j+1} - b_j = a_j$, then we can write down the answer immediately, that is $b_{n+1} - b_1$.

(i) Let $b_j = j^2$ (1)

then
$$b_{j+1} - b_j = (j+1)^2 - j^2$$
$$= 2j + 1$$

thus here, we take $a_j = 2j + 1$

Now using $\sum_{j=1}^{n} a_j = b_{n+1} - b_1$

$$\sum_{j=1}^{n} (2j+1) = (n+1)^2 - 1^2 \quad \text{by (1)}$$

$$2\sum_{j=1}^{n} j+\sum_{j=1}^{n} 1=n^2+2n$$

$$2\sum_{j=1}^{n} j+n=n^2+2n$$

$$2\sum_{j=1}^{n} j=n^2+2n-n$$

$$2\sum_{j=1}^{n} j=n^2+n$$

Hence $\displaystyle\sum_{j=1}^{n} j=\frac{n(n+1)}{2}$

(ii) let $b_j = j^3$ (2)

then $b_{j+1}-b_j=(j+1)^3-j^3$

$$=3j^2+3j+1$$

thus here $a_j=3j^2+3j+1$.

Now, using the following $\displaystyle\sum_{j=1}^{n} a_j = b_{n+1}-b_1$

$$\sum_{j=1}^{n}(3j^2+3j+1) = (n+1)^3-1^3 \quad \text{by (2)}$$

$$3\sum_{j=1}^{n} j^2 + 3\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 = (n+1)^3-1$$

$$3\sum_{j=1}^{n} j^2+3\left(\frac{n(n+1)}{2}\right)+n=(n+1)^3-1$$

$$3\sum_{j=1}^{n} j^2=(n+1)^3-1-3\left(\frac{n(n+1)}{2}\right)-n$$

$$=(n+1)^3-(n+1)-\frac{3}{2}n(n+1)$$

$$=\frac{n+1}{2}\left[2(n+1)^2-2-3n\right]$$

$$=\frac{n+1}{2}\left[2n^2+2+4n-2-3n\right] = \frac{n+1}{2}\left[2n^2+n\right]$$

$$=\frac{n(n+1)(2n+1)}{2}$$

Hence $$\sum_{j=1}^{n} j^2 = \frac{1}{6}[n(n+1)(2n+1)]$$

(iii) Let $b_j = j^4$ (3)

then $b_{j+1} - b_j = (j+1)^4 - j^4 = 4j^3 + 6j^2 + 4j + 1$

we take $a_j = 4j^3 + 6j^2 + 4j + 1$

Now, using the following

$$\sum_{j=1}^{n} a_j = b_{n+1} - b_1$$

$$\sum_{j=1}^{n} (4j^3 + 6j^2 4j + 1) = (n+1)^4 - 1^4 \text{ by(1)}$$

$$4\sum_{j=1}^{n} j^3 + 6\sum_{j=1}^{n} j^2 + 4\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 = (n+1)^4 - 1$$

$$4\sum_{j=1}^{n} j^3 + 6\left(\frac{n(n+1)(2n+1)}{6}\right) + 4\left(\frac{n(n+1)}{2}\right) + n = (n+1)^4 - 1$$

$$4\sum_{j=1}^{n} j^3 = (n+1)^4 - 1 - n(n+1)(2n+1) - 2n(n+1) - n$$

$$= (n+1)^4 - (n+1) - n(n+1)(2n+1) - 2n(n+1)$$

$$= (n+1)[(n+1)^3 - 1 - 2n^2 - n - 2n]$$

$$= (n+1)[n^3 + 3n^2 + 3n + 1 - 1 - 2n^2 - 3n] = (n+1)\ [n^3 + n^2]$$

$$= n^2(n+1)^2$$

Hence $$\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 3: Find the sum of the n terms of the series.

$$1\cdot2 + 2\cdot3 + 3\cdot4 + \cdots$$

Solution: Let T_j be the general term of the given series, then

$$T_j = j(j+1)$$

and $$\sum_{j=1}^{n} T_j = \sum_{j=1}^{n} (j^2 + j)$$

$$=\sum_{j=1}^{n} j^2+\sum_{j=1}^{n} j$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{6}\ [2n+1+3]$$

$$=\frac{n(n+1)}{6}\ (2n+4)$$

$$=\frac{n(n+1)(n+2)}{3}$$

Example 4: Find the sum of the n terms of the series

$$1\cdot 2^2+2\cdot 3^2+3\cdot 4^2+\ldots$$

Solution: Here $T_j=j(j+1)^2$

then $\displaystyle\sum_{j=1}^{n} T_j=\sum_{j=1}^{n}(j^3+2j^2+j)$

$$=\sum_{j=1}^{n} j^3+2\sum_{j=1}^{n} j^2+\sum_{j=1}^{n} j$$

$$=\frac{n^2(n+1)^2}{4}+2\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{12}\ [3n^2+3n+8n+4+6]$$

$$=\frac{n(n+1)}{12}\ [3n^2+11n+10]$$

$$=\frac{1}{12}\ n(n+1)(n+2)(3n+5)$$

Example 5: Find the sum of n terms of the series whose nth term is $2^{n-1}+8n^3-6n^2$.

Solution: Given that $T_n=2^{n-1}+8n^3-6n^2$

then $T_j=2^{j-1}+8j^3-6j^2$

$$\sum_{j=1}^{n} T_j=\sum_{i=1}^{n}(2^{j-1}+8j^3-6j^2)$$

$$=\sum_{j=1}^{n} 2^{j-1}+8\sum_{j=1}^{n} j^3-6\sum_{j=1}^{n} j^2$$

$$=\left[\frac{1(2^n-1)}{2-1}\right]+8\left[\frac{n^2(n+1)^2}{4}\right]-6\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$=2^n-1+n(n+1)[2n^2+2n-2n-1]$$

$$=2^n-1+n(n+1)(2n^2-1)$$

EXERCISE 5.1

1. **Sum the following series up to n terms**

   (i) $1^2+3^2+5^2+7^2+\cdots$ (ii) $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\cdots$

   (iii) $2^2+4^2+6^2+\cdots$ (iv) $1^3+3^3+5^3+\cdots$ (v) $1^3+5^3+9^3+\cdots$

2. Find the sum $1\cdot2+2\cdot3+3\cdot4+\cdots+99\cdot100$
3. Find the sum $1^2+3^2+5^2+7^2+\cdots+99^2$
4. Find the sum $2+(2+5)+(2+5+8)+\cdots$ to n terms
5. Sum $2+5+10+17+\cdots$ to n terms
6. Sum to n terms. $1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\cdots$
7. Sum to n terms $1\cdot5\cdot9+2\cdot6\cdot10+3\cdot7\cdot11+\cdots$
8. Find the sum to 2n terms of the series whose nth term is $4n^2+5n+1$
9. Find the sum of n terms of the series whose nth term is:

   (i) $n^2(2n+3)$ (ii) $3(4^n+2n^2)-4n^3$

5.2 Arithmetico-Geometric series

Since we are already familiar with the arithmetic and geometric sequences and their related series. Now, we discuss here another important sequence and its related series, which we obtain from arithmetic and geometric sequences.

5.2.1 A series which is obtained by multiplying the corresponding terms of an arithmetic series and a geometric series is called **Arithmetico–Geometric series.**

For example,

$$[a+(a+d)+(a+2d)+\ldots+(a+(n-1)d]\,[1+r+r^2+\ldots+r^{n-1}]$$

$$=a+(a+d)r+(a+2d)r^2+\ldots+(a+(n-1)d)\ r^{n-1}$$

which is arithmetico–geometric series.

nth term of Arithmetico-Geometric Series

A series which is formed by multiplying the corresponding terms of an A. P. and a G. P. is called an arithmetico-geometric series. Thus nth term of such series has the form $[a+(n-1)d]\times r^{n-1}$

5.2.2 Sum of n terms of Arithmetico-Geometric Series

Let $S_n = a+(a+d)r+(a+2d)r^2+...+[a+(n-1)d]\ r^{n-1}$ (1)

then $rS_n = ar+(a+d)r^2+...+[a+(n-2)d]\ r^{n-1}+[a+(n-1)d]\ r^n$ (2)

subtracting (2) from (1) we obtain

$$(1-r)S_n = a+(dr+dr^2+...+dr^{n-1})-[a+(n-1)d]\ r^n$$

$$\therefore S_n = \frac{a}{1-r}+\frac{1}{1-r}\left[\frac{dr(1-r^{n-1})}{1-r}\right]-\frac{1}{1-r}[a+(n-1)d]\ r^n$$

$$=\frac{a}{1-r}+\frac{dr}{(1-r)^2}-\frac{dr^n}{(1-r)^2}-\frac{[a+(n-1)d]r^n}{(1-r)} \quad (3)$$

which is the required sum of the n terms of arithmetico–geometric series.

Example 6: Sum the series $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+.....$ to n terms.

Solution: Let $S=1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+.....+\frac{3n-2}{5^{n-1}}$

$$\therefore \frac{1}{5}S=\frac{1}{5}+\frac{4}{5^2}+\frac{7}{5^3}+.....+\frac{3n-5}{5^{n-1}}+\frac{3n-2}{5^n}$$

$$\therefore \frac{4}{5}S=1+\left(\frac{3}{5}+\frac{3}{5^2}+\frac{3}{5^3}+.........+\frac{3}{5^{n-1}}\right)-\frac{3n-2}{5^n}$$

$$=1+\frac{3}{5}\left(1+\frac{1}{5}+\frac{1}{5^2}+.........+\frac{1}{5^{n-2}}\right)-\frac{3n-2}{5^n}$$

$$=1+\frac{3}{5}\left(\frac{1-\frac{1}{5^{n-1}}}{1-\frac{1}{5}}\right)-\frac{3n-2}{5^n}$$

> **Note**
>
> **Sum to infinity of an Arithmetico–Geometric Series**
>
> Let $|r|<1$
>
> Then $r^n\to 0$ as $n\to\infty$
>
> $\therefore$ Equation (3) reduces to
>
> $$S_\infty=\frac{a}{1-r}+\frac{dr}{(1-r)^2}$$
>
> which is the required sum to infinity of arithmetico–geometric series

$$=1+\frac{3}{4}\left(1-\frac{1}{5^{n-1}}\right)-\frac{3n-2}{5^n}=1+\frac{3}{4}-\frac{3}{4\cdot 5^{n-1}}-\frac{3n-2}{5^n}=\frac{7}{4}-\frac{1}{5^{n-1}}\left(\frac{3}{4}+\frac{3n-2}{5}\right)$$

$$=\frac{7}{4}-\frac{1}{5^{n-1}}\left(\frac{15+12n-8}{4\cdot 5}\right)=\frac{7}{4}-\frac{12n+7}{4.5^n} \qquad \therefore S=\frac{35}{16}-\frac{12n+7}{16.5^{n-1}}$$

Example 7: Sum the series.

$$2\cdot1+4\cdot3+6\cdot9+8\cdot27+10\cdot81+\ldots\text{to n terms.}$$

Solution: Let $S=2\cdot1+4\cdot3+6\cdot9+\ldots+(2n-2)\cdot3^{n-2}+2n\cdot3^{n-1}$. (i)

Multiplying by 3, the common ratio of the geometric series, we get

$3\cdot S=2\cdot3+4\cdot9+6\cdot27+\ldots+(2n-2)\cdot3^{n-1}+2n\cdot3^n$. (ii)

Subtracting (ii) from (i), we get

$$(1-3)S=2\cdot1+\{3(4-2)+9(6-4)+27(8-6)+\ldots+3^{n-1}(2n-2n-2)\}-2n\cdot3^n.$$

$$\therefore(-2)\cdot S=2\cdot1+\{2(3+9+27+\ldots\text{to}(n\text{-}1)\text{ terms})\}-2n\cdot3^n$$

$$=2+2\left[3\cdot\frac{(3^{n-1}-1)}{3-1}\right]-2n\cdot3^n$$

$$=2+3^n-3-2n\cdot3^n=-1-3^n(2n-1).$$

$$\therefore\ S=\frac{1}{2}\left[1+3^n(2n-1)\right]$$

Example 8: If $x<1$, sum the series

$1+2x+3x^2+4x^3+\ldots\ldots$ to infinity

Solution: Let $S=1+2x+3x^2+4x^3+\ldots\ldots$ (i)

$\therefore$ $xS=x+2x^2+3x^3+\ldots\ldots$ (ii)

Subtracting (ii) from (i), we get

$\therefore S(1-x)=1+x+x^2+x^3+\ldots\ldots$

The R·H·S is an infinite geometric series with $a_1=1$ and $r=x<1$

$$S\,(1-x)=\frac{1}{1-x}$$

$$\therefore S=\frac{1}{(1-x)^2}$$

Example 9: Show that $2^{\frac{1}{4}} \times 4^{\frac{1}{8}} \times 8^{\frac{1}{16}} \times 16^{\frac{1}{32}} \times \ldots \infty = 2$

Solution: Let $x = 2^{\frac{1}{4}} \times 4^{\frac{1}{8}} \times 8^{\frac{1}{16}} \times 16^{\frac{1}{32}} \times \ldots \infty$

$$\log x = \log 2^{\frac{1}{4}} + \log 4^{\frac{1}{8}} + \log 8^{\frac{1}{16}} + \log 16^{\frac{1}{32}} + \ldots \infty$$

$$\log x = \frac{1}{4}\log 2 + \frac{1}{8}\log 4 + \frac{1}{16}\log 8 + \frac{1}{32}\log 16 + \ldots \infty$$

$$\log x = \frac{1}{4}\log 2 + \frac{1}{8}\log 2^2 + \frac{1}{16}\log 2^3 + \frac{1}{32}\log 2^4 + \ldots \infty$$

$$\log x = \frac{1}{4}\log 2 + \frac{2}{8}\log 2 + \frac{3}{16}\log 2 + \frac{4}{32}\log 2 + \ldots \infty$$

$$\log x = \left(\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \ldots \infty\right)\log 2 \qquad \text{(i)}$$

Now, $\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \ldots \infty$ is an arithmetico-geometric series

Let $S = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \ldots \infty \qquad \text{(ii)}$

$$\frac{1}{2}S = \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{4}{64} + \ldots \infty \qquad \text{(iii)}$$

On subtracting Eq(iii) from Eq(ii), we get

$$\frac{1}{2}S = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \infty$$

$$\Rightarrow \frac{1}{2}S = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots \infty$$

$$\frac{1}{2}S = \frac{\frac{1}{2^2}}{1 - \frac{1}{2}}$$

$$\frac{1}{2}S = \frac{1}{2}$$

$$S = 1$$

$$\therefore (i) \Rightarrow \log x = 1 \times \log 2$$

$$\log x = \log 2$$

$$\therefore x = 2.$$

EXERCISE 5.2

1. Sum to n terms the following series

(i) $1\cdot2+2\cdot2^2+3\cdot2^3+4\cdot2^4+\ldots$

(ii) $1+4x+7x^2+10x^3+\ldots$

(iii) $1+2x+3x^2+4x^3+\ldots$

(iv) $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots$

(v) $1-7x+13x^2-19x^3+\ldots$

2. Find the sum to infinity of the following series

(i) $1^2+3^2x+5^2x^2+7^2x^3+\ldots, x<1$

(ii) $1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\frac{25}{3^4}+\ldots$

3. Find the nth term of the following arithmetico–geometric series

$\frac{0}{1}+\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\ldots$

4. Find the sum of the following Arithmetico–geometric series

$5+\frac{7}{3}+1+\frac{11}{27}+\ldots$

5. If the sum to infinity of the series $3+5r+7r^2+\ldots\ldots\infty$ is $\frac{44}{9}$, **find the value of** r.

5.3 The Method of Differences

In the case of some series in which the difference of successive terms form an A.P, or G.P., the following method can be employed to find the nth term. The sum of such a series to n terms may then be obtained.

Example 10: Find the nth term and the sum to n terms of the series

$$1+7+17+31+49+\ldots\ldots\ldots\ldots$$

Solution:

We have

$$a_2-a_1=6$$
$$a_3-a_2=10$$
$$a_4-a_3=14$$
$$\ldots\quad\ldots\quad\ldots$$
$$\ldots\quad\ldots\quad\ldots$$
$$a_n-a_{n-1}=\text{(n–1)th term of the sequence } 6,10,14,\ldots$$

Adding column-wise, we get

$$a_n-a_1=6+10+14+18+\ldots\ldots to\,(n-1)\,terms,$$

$$a_n - a_1 = \frac{n-1}{2}[2\cdot 6 + (n-2)\cdot 4]$$

$$a_n - a_1 = \frac{n-1}{2}[12+4n-8] = \frac{n-1}{2}[4n+4] = (n-1)(2n+2) \Rightarrow a_n = a_1 + (n-1)(2n-2)$$

$$\therefore a_n = 1 + 2(n-1)(n+1) \qquad [\because a_1 = 1]$$

$$\therefore a_n = 2n^2 - 1$$

$$\therefore a_r = 2r^2 - 1$$

$$\therefore \sum_1^n a_r = 2\sum_1^n r^2 - \sum_1^n 1 = 2\cdot\frac{n(n+1)(2n+1)}{6} - n = \frac{n(n+1)(2n+1)}{3} - n$$

$$= \frac{n(n+1)(2n+1) - 3n}{3} = \frac{n(n+2)(2n-1)}{3}$$

$\therefore$ the required sum $= \dfrac{n(n+2)(2n-1)}{3}$

Example 11: Find the nth term and the sum to n terms of the series
$3 + 5 + 9 + 17 + 31 + \ldots\ldots\ldots\ldots$

Solution:

$$a_2 - a_1 = 2$$

$$a_3 - a_2 = 4$$

We have

$$a_4 - a_3 = 8$$

$$\ldots \quad \ldots \quad \ldots$$

$$a_n - a_{n-1} = \text{(n-1)th term of the sequence } 2, 4, 8, \ldots$$

Which is a G.P. Adding column-wise, we get

$$a_n - a_1 = 2 + 4 + 8 + \ldots\ldots to\ (n-1)\ terms,$$

$$a_n - a_1 = \frac{2(2^{n-1}-1)}{2-1} = 2^n - 2$$

$$\therefore a_n = 2^n - 2 + 3 \qquad [\because a_1 = 3]$$

$$\therefore a_n = 2^n + 1$$

$$\therefore a_r = 2^r + 1$$

$$\therefore \sum_1^n a_r = [2 + 2^2 + 2^3 + 2^4 + \ldots\ldots + 2^n] + \sum 1$$

$$= \frac{2(2^n - 1)}{2-1} + n = 2^{n+1} + n - 2$$

$\therefore$ the required sum $= 2^{n+1} + n - 2.$

EXERCISE 5.3

Find the nth term and the sum to n terms of each of the following series:

1. $4+13+28+49+76+\ldots\ldots$
2. $4+14+30+52+80+114+\ldots\ldots$
3. $4+10+18+28+40+\ldots\ldots$
4. $3+5+11+29+83+245+\ldots\ldots$
5. $3+9+21+45+93+189+\ldots\ldots$
6. $28+32+52+152+652+\ldots\ldots$

5.4 Summation by the method of Partial Fractions

If the general term of a series consists of the products of the reciprocals of two or more consecutive terms of an A.P., then the term can be split up into partial fractions and the series can be summed. The method is illustrated in the following examples.

Example 12: Sum the series $\frac{1}{3\cdot7}+\frac{1}{7\cdot11}+\frac{1}{11\cdot15}+\frac{1}{15\cdot19}+\cdots$ *to n terms*

Solution: Here, the factors in the denominators are the products of two successive terms of an A. P 3, 7, 11, 15, 19,

$\therefore$ *rth* term of the given series, $a_r=\frac{1}{(4r-1)(4r+3)}$

Expressing a_r as the difference of its partial fractions, we have

$$a_r=\frac{1}{4}\left[\frac{1}{4r-1}-\frac{1}{4r+3}\right]$$

By putting $r=1,2,3,\ldots..(n-1),n$ in succession, we get

$$a_1=\frac{1}{4}\left[\frac{1}{3}-\frac{1}{7}\right]$$

$$a_2=\frac{1}{4}\left[\frac{1}{7}-\frac{1}{11}\right]$$

$$a_3=\frac{1}{4}\left[\frac{1}{11}-\frac{1}{15}\right]$$

... ...

... ...

... ...

$$a_{n-1} = \frac{1}{4}\left[\frac{1}{4n-5} - \frac{1}{4n-1}\right]$$

$$a_n = \frac{1}{4}\left[\frac{1}{4n-1} - \frac{1}{4n+3}\right]$$

Adding column-wise, we get

$$\sum_{1}^{n} a_r = \frac{1}{4}\left[\frac{1}{3} - \frac{1}{4n+3}\right] = \frac{n}{3(4n+3)}$$

$\therefore$ The required sum $= \dfrac{n}{3(4n+3)}$.

Example 13: Find the sum of the series:

$$\frac{1}{1\cdot4} + \frac{1}{4\cdot7} + \frac{1}{7\cdot10} + \ldots \text{ to infinity.}$$

Solution: Here $T_n = \dfrac{1}{(3n-2)(3n+1)}$

Breaking it into Partial Fractions, we have

$$\frac{1}{(3n-2)(3n+1)} = \frac{A}{3n-2} + \frac{B}{3n+1}$$

Multiplying both sides by $(3n-2)(3n+1)$, we have

$$1 = A(3n+1) + B(3n-2)$$

Comparing the coefficient of n and the constants both sides, we get

$$0 = 3A + 3B \qquad \text{(i)}$$

$$1 = A - 2B \qquad \text{(ii)}$$

Solving (i) and (ii) we get $A = \frac{1}{3}$, $B = -\frac{1}{3}$

$$\therefore \quad T_n = \frac{1}{3(3n-2)} - \frac{1}{3(3n+1)} = \frac{1}{3}\left[\frac{1}{3n-2} - \frac{1}{3n+1}\right]$$

and $\displaystyle\sum_{k=1}^{\infty} T_k = \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{1}{3k-2} - \frac{1}{3k+1}\right)$

$$= \frac{1}{3}\left[\left(1-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{7}\right) + \left(\frac{1}{7}-\frac{1}{10}\right) + \ldots\right] = \frac{1}{3}(1) = \frac{1}{3}$$

EXERCISE 5.4

1. Find the sum of the following:

(i) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots$ to n terms.

(ii) $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots$ to n terms.

(iii) $\frac{1}{2\cdot 5}+\frac{1}{5\cdot 8}+\frac{1}{8\cdot 11}+\ldots$ to infinity.

(iv) $\frac{1}{4\cdot 13}+\frac{1}{13\cdot 22}+\frac{1}{22\cdot 31}+\ldots$ to infinity.

2. Find sum of the series: $\sum_{k=1}^{n}\frac{1}{9k^2+3k-2}$

3. Find sum of the series: $\sum_{k=2}^{n}\frac{1}{(k^2-k)}$

4. Find sum of the series: $\sum_{k=1}^{n}\frac{1}{k^2+7k+12}$

REVIEW EXERCISE 5

1. Choose the correct option

(i) If $t_n = 6n + 5$, then $t_{n+1} =$

(a) 6n–1 (b) 6n +11 (c) 6n + 6 (d) 6n–5

(ii) The sum to infinity of the series $1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\cdots$

(a) 6 (b) 2 (c) 3 (d) 4

(iii) Sum the series: $1 + 2\cdot 2 + 3\cdot 2^2 + \ldots\ldots + 100\cdot 2^{99}$

(a) $99\cdot 2^{100}$ (b) $100\cdot 2^{100}$ (c) $99\cdot 2^{100}+1$ (d) $1000\cdot 2^{100}$

(iv) The nth term of the series $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots$ is

(a) $(n^2 - n)$ (b) $(n^2 + n)$ (c) n^2 (d) None of these

(v) The sum of n terms of the series whose nth term is $1+2^n$

(a) $n + 2^{n-1}$ (b) $(n + 1) + 2^{n+1}$ (c) $n + 2(2^n - 1)$ (d) None of these

(vi) Evaluate $\sum(3 + 2^r)$, where r = 1, 2, 3, …. 10

(a) 2051 (b) 2049 (c) 2076 (d) 1052

(vii) What is the nth term of the series $1+\frac{(1+2)}{2}+\frac{(1+2+3)}{3}+\ldots$?

(a) $\frac{n+1}{2}$ (b) $\frac{n(n+1)}{2}$ (c) $n^2-(n+1)$ (d) $\frac{(n+1)(2n+3)}{2}$

(viii) Sum of n terms of the series $1^3 + 3^3 + 5^3 + 7^3 + \ldots$ is

(a) $n^2(2n^2 - 1)$ (b) $2n^3 + 3n^2$ (c) $n^3(n-1)$ (d) $n^3 + 8n + 4$

2. Sum the series to n terms $1\cdot2+2\cdot3+3\cdot4+\cdots$

3. Sum the series $1\cdot3\cdot5+2\cdot4\cdot6+3\cdot5\cdot7+$ to n terms.

4. Sum the series $\dfrac{1}{1\cdot4\cdot7}+\dfrac{1}{4\cdot7\cdot10}+\dfrac{1}{7\cdot10\cdot13}+\ldots$

5. Sum the series $5+12x+19x^2+26x^3+\cdots$ to n terms.

6. Sum the series: $\dfrac{1}{1\cdot2}+\dfrac{1}{2\cdot3}+\dfrac{1}{3\cdot4}+\ldots$ to n terms.

7. Find the sum of n terms of the series

(i) Sum the series: $1\cdot2^2+3\cdot3^2+5\cdot4^2+\cdots$ to n terms.

(ii) Sum the series: $3\cdot1^2+5\cdot2^2+7\cdot3^2+\cdots$ to n terms.

8. Find the sum of *n* terms of the series whose *n*th term is

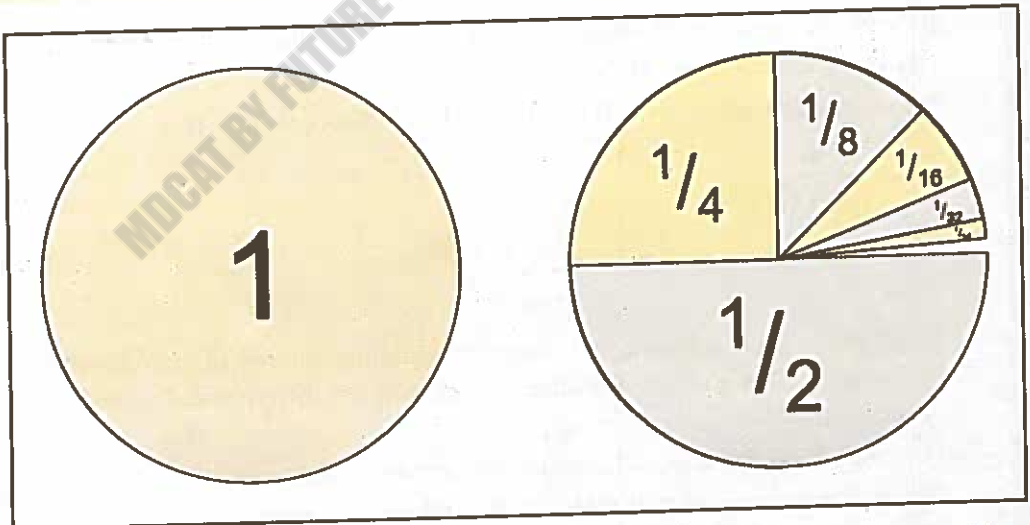
(i) n^3+3^n (ii) $2n^2+3n$ (iii) $n(n+1)(n+4)$ (iv) $(2n-1)^2$

9. Find the sum of the first n terms of the series

(i) $3+7+13+21+31+\cdots$ (ii) $2+5+14+41+\cdots$

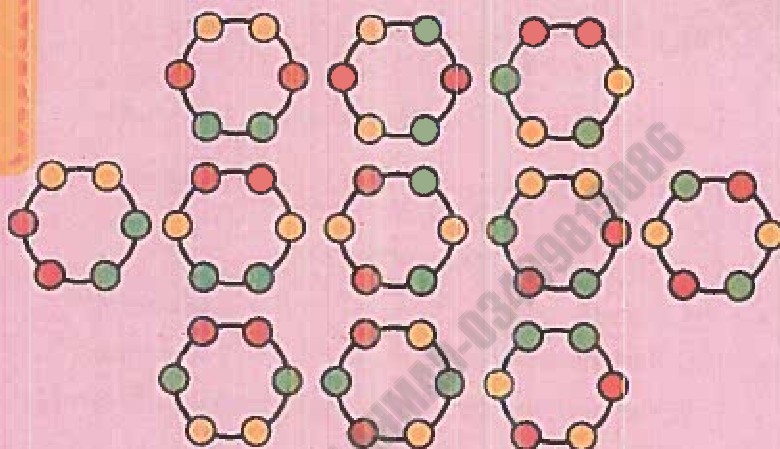
10. Find the n^{th} term and the sum to n terms of the series

$1+(1+\frac{1}{2})+(1+\frac{1}{2}+\frac{1}{4})+(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8})+\cdots$



UNIT 6

PERMUTATION, COMBINATION AND PROBABILITY



STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Know Kramp's factorial notation to express the product of first n natural numbers by n!.
- Recognize the fundamental principle of counting and illustrate this principle using tree diagram.
- Explain the meaning of permutation of n different objects taken r at a time and know the notation ^nP_r.
- Prove that $^nP_r = n(n-1)(n-2) \ldots (n-r+1)$ and hence deduce that

  $$^nP_r = \frac{n!}{(n-r)!},$$

- $^nP_n = n!$
- $0! = 1$.
- Apply ^nP_r to solve relevant problems of finding the number of arrangements of n objects taken r at a time (when all n objects are different and when some of them are alike).
- Find the arrangement of different objects around a circle.
- Define combination of n different objects taken r at a time.

STUDENTS LEARNING OUTCOMES

- Prove the $^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ and deduce that
  - $\binom{n}{n} = \binom{n}{0} = 1$
  - $\binom{n}{r} = \binom{n}{n-r}, \binom{n}{1} = \binom{n}{n-1} = n,$
  - $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$
- Solve problems involving combination.
- Define the following:
  - statistical experiment,
  - sample space and an event,
  - mutually exclusive events,
  - equally likely events,
  - dependent and independent events,
  - simple and compound events.
- Recognize the formula for probability of occurrence of an event E, that is $P(E) = \frac{n(E)}{n(S)}$, $0 \le P(E) \le 1$
- Apply the formula for finding probability in simple cases.
- Use Venn diagrams and tree diagrams to find the probability for the occurrence of an event.
- Define the conditional probability
- Recognize the addition theorem (or law) of probability $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, where A and B are mutually exclusive events.
- Recognize multiplication theorem (or law) of probability $P(A \cap B) = P(A)\,P(B|A)$ or $P(A \cap B) = P(B)\,P(A|B)$ where $P(B|A)$ and $P(A|B)$ are conditional probabilities.
  Deduce that $P(A \cap B) = P(A)\,P(B)$ where A and B are independent events.
- Use theorem of addition and multiplication of probability to solve related problems

6.1 Introduction

Counting is one of the most fundamental skills. People start to count on their fingers when they are in kindergarten or even earlier. But how to count quickly, correctly, and systematically is a lifelong course.
In order to study probability, it is first necessary to learn about combinatorics, the theory of counting.

In this unit, we will develop techniques and formulae for counting the number of objects in a set. These formulae are used in computer science to analyze algorithms. They are also used to determine probabilities, the likelihood that a certain outcome of a random experiment will occur.

6.1.1 Kramp's Factorial notation to express the product of first n natural numbers by n!

Factorial Notation

If n is a positive integer, the notation $n!$ (read "n factorial") is the product of all positive integers from n down through 1.

$n! = n(n-1)(n-2)\text{--------}(3)(2)(1)$

$0!$ (zero factorial), by definition, $0! = 1$

Remember

The First Ten Factorials

$0! = 1$
$1! = 1$
$2! = 2 \cdot 1 = 2$
$3! = 3 \cdot 2 \cdot 1 = 6$
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$
$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040$
$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$
$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362{,}880$

Technology

Most calculators have factorial keys. To find 5!, most calculators use one of the following:

Many Scientific Calculators

5 [x!]

Many Graphing Calculators

5 [!] [ENTER]

Note The Difference

$2 \cdot 3! = 2(3 \cdot 2 \cdot 1) = 12$

$(2 \cdot 3)! = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

Example 1: Simplify the following expressions:

$$a.\frac{8!}{7!} \quad b.\frac{5!}{2!.3!} \quad c.\frac{1}{2!.4!}+\frac{1}{3!.3!} \quad d.\frac{(n+1)!}{n!} \quad e.\frac{n!}{(n-3)!}$$

Solution: a. $\frac{8!}{7!}=\frac{8.7!}{7!}=8$

b. $\frac{5!}{2!\cdot 3!}=\frac{5.4.3!}{2!\cdot 3!}=\frac{5.4}{2\cdot 1}=10$

c. $\frac{1}{2!\cdot 4!}+\frac{1}{3!\cdot 3!}=\frac{1}{2!\cdot 4!}.\frac{3}{3}+\frac{1}{3!\cdot 3!}.\frac{4}{4}=\frac{3}{3\cdot 2!\cdot 4!}+\frac{4}{3!\cdot 4\cdot 3!}$

$$=\frac{3}{3!\cdot 4!}+\frac{4}{3!\cdot 4!}=\frac{7}{3!\cdot 4!}=\frac{7}{144}$$

d. $\frac{(n+1)!}{n!}=\frac{(n+1).n!}{n!}=n+1$

e. $\frac{n!}{(n-3)!}=\frac{n.(n-1).(n-2).(n-3)!}{(n-3)!}$

$$=n.(n-1).(n-2)=n^3-3n^2+2n$$

Practice

Evaluate each factorial expression:

a. $\frac{14!}{2!12!}$ b. $\frac{n}{(n-1)!}$

Example 2: Write the following in factorial form:

(i) $\frac{13\cdot 17}{9\cdot 8\cdot 7\cdot 5}$ (ii) $\frac{(n-3)(n-2)(n-1)}{n(n-4)}$

Solution: (i) $\frac{13\cdot 17}{9\cdot 8\cdot 7\cdot 5}=\frac{17\cdot 16\cdot 15\cdot 14\cdot 13\cdot 12!}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4!}.\frac{13\cdot 6\cdot 4!}{16\cdot 15\cdot 14\cdot 13\cdot 12!}$

$$=\frac{17!}{9!}.\frac{6\cdot 5!\cdot 4!\cdot 13\cdot 12!}{16!\cdot 5!\cdot 12!}=\frac{17!\cdot 13!\cdot 6!\cdot 4!}{16!\cdot 12!\cdot 9!\cdot 5!}$$

(ii) $\frac{(n-3)(n-2)(n-1)}{n(n-4)}=\frac{(n-1)(n-2)(n-3)}{n(n-4)}=\frac{(n-1)(n-2)(n-3)(n-4)!}{n(n-4)(n-4)!}$

$$=\frac{(n-1)!(n-1)(n-2)(n-3)(n-5)!}{n(n-1)(n-2)(n-3)(n-4)(n-5)!(n-4)!}$$

$$=\frac{(n-1)!(n-5)!}{n!(n-4)!}\cdot\frac{(n-1)(n-2)(n-3)(n-4)!}{(n-4)!}$$

$$=\frac{(n-1)!(n-5)!}{n!(n-4)!}\cdot\frac{(n-1)!}{(n-4)!}=\frac{((n-1)!)^2(n-5)!}{n!((n-4)!)^2}$$

EXERCISE 6.1

1. Evaluate the following.

(i) $\dfrac{10!}{3!3!4!}$ (ii) $\dfrac{3!+4!}{5!-4!}$ (iii) $\dfrac{(n-1)!}{(n+1)!}$ (iv) $\dfrac{10!}{(5!)^2}$

2. Write the following in terms of factorials.

(i) $19\cdot18\cdot17\cdot16\cdot15\cdot14$ (ii) $2\cdot4\cdot6\cdot8\cdot10\cdot12$

(iii) $n\left(n^2-1\right)$ (iv) $\dfrac{n(n+1)(n+2)}{3}$

3. Prove the following.

(i) $\dfrac{1}{6!}+\dfrac{2}{7!}+\dfrac{3}{8!}=\dfrac{75}{8!}$ (ii) $\dfrac{(n+5)!}{(n+3)!}=n^2+9n+20$

4. Find the value of n, when

(i) $\dfrac{n(n!)}{(n-5)!}=\dfrac{12(n!)}{(n-4)!}$ (ii) $\dfrac{n!}{(n-4)!}:\dfrac{(n-1)!}{(n-4)!}=9:1$

5. Show that (i) $\dfrac{(2n)!}{n!}=2^n\left(1\cdot3\cdot5\cdots(2n-1)\right)$

(ii) $\dfrac{(2n+1)!}{n!}=2^n\left(1\cdot3\cdot5\cdots(2n-1)(2n+1)\right)$

6.2 Permutation

As we know that counting plays a vital role in many areas, such as probability, statistics and computer science. In this section and in the next, we shall look at special types of counting problems and develop general formulae for solving them.

The following principle of counting will be helpful and basic to all our work.

6.2.1 Fundamental Principle of Counting

Let $E_1, E_2, \ldots, E_k$ be a sequence of k events. If for each i, E_i can occur in m_i different ways, then the total number of ways the events may take place is the product $m_1 m_2 \ldots m_k$.

This principle is also known as the multiplication principle.

Example 3: How many different 6-place vehicle number plates are possible if the first 3 places are to be occupied by letters and the final 3 by numbers?

Solution: Since the first three places are to be occupied by the letters A,B,C,...,Z and the final 3 places by the numbers 0, 1, 2, . . . , 9.

Hence each event E_i, $i = 1, 2, 3$ occurs in $m_i = 26$, $i = 1, 2, 3$ different ways and each E_i, $i = 4, 5, 6$ occurs in $m_i = 10$, $i = 4, 5, 6$ different ways. Then by the fundamental counting principle the total number of vehicle number plates is

$$m_1 \cdot m_2 \cdot m_3 \cdot m_4 \cdot m_5 \cdot m_6 = 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17576000$$

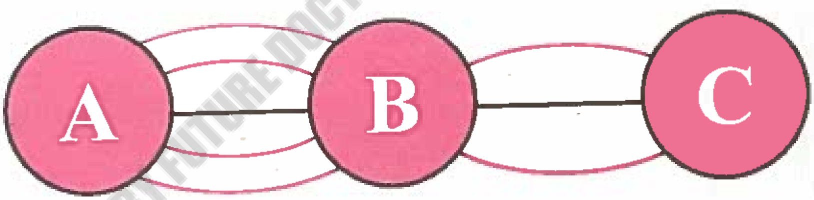
Example 4: How many functions defined on n points is possible if each functional value is either 0 or 1?

Solution: Let the points be 1, 2, 3, . . . , n.

Since $f(i) = 0$ or 1 for each $i = 1, 2, 3, \ldots, n$. Hence each event E_i, $i = 1, 2, 3, \ldots, n$ has $m_i = 2$, $i = 1, 2, 3, \ldots, n$ possibilities. Thus by the fundamental counting principle the total numbers of possible functions is

$$m_1 \cdot m_2 \cdot m_3 \cdots m_n = 2 \cdot 2 \cdot 2 \cdots 2 = 2^n$$

Example 5: There are 5 roads joining A to B and 3 roads joining B to C. Find how many different routes there are from A to C via B.



Solution: There are two operations to be performed in succession.

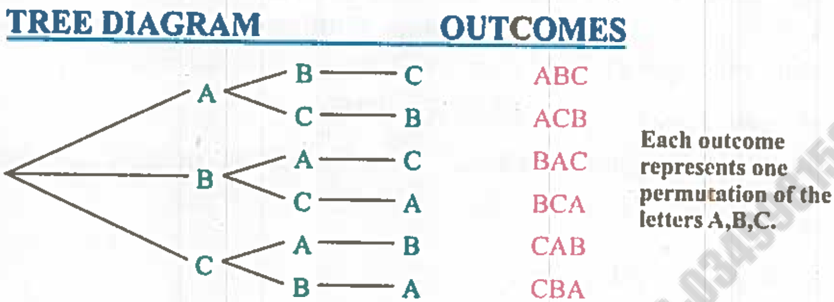
A to B	5 ways
B to C	3 ways

Number of routes from A to C $= 5 \times 3 = 15$

Example 6: How many 3-letter code symbols can be formed with the letters A, B, C without repetition?

Solution: Consider placing the letters in these boxes. | | | |

We can select any of the 3 letters for the first letter in the symbol. Once this letter has been selected, the second must be selected from the 2 remaining letters. After this, the third letter is already determined, since only 1 possibility is left. That is, we can place any of the 3 letters in the first box, either of the remaining 2 letters in the second box, and the only remaining letter in the third box. The possibilities can be arrived at using a tree diagram, as shown below.



We see that there are 6 possibilities. The set of all the possibilities is {ABC, ACB, BAC, BCA, CAB, CBA}.

Example 7: How many 3-letter code symbols can be formed with the letters A, B, C, D, and E with repetition (that is, allowing letters to be repeated)?

Solution: Since repetition is allowed, there are 5 choices for the first letter, 5 choices for the second, and 5 for the third. Thus there are $5 \cdot 5 \cdot 5$, or 125 code symbols.

Example 8: How many 5-letter code symbols can be formed with the letters A, B, C, and D if we allow a letter to occur more than once?

Solution: We can select each of the 5 letters in 4 ways. That is, we can select the first letter in 4 ways, the second in 4 ways, and so on. Thus there are 4^5, or 1024 arrangements.

6.2.2 Explaining the meaning of permutation

An ordered arrangement of a finite number of elements taken some or all at a time is called a **permutation** of these elements.

We use the notation ^nP_r or $P(n,r)$ to denote the number of permutations of n elements taken r at a time, where r is a positive integer such that $r \le n$.

Now, we develop general formula for the solution of special types of counting problems.

6.2.3 $^nP_r = n(n-1)(n-2)...(n-r+1)$

Theorem: Prove that $^nP_r = n(n-1)(n-2)...(n-r+1)$ and hence deduce the following: (i) $^nP_r = \frac{n!}{(n-r)!}$ (ii) $^nP_n = n!$ (iii) $0! = 1$

Proof: To find a formula for ^nP_r, we note that the task of obtaining an ordered arrangement of n elements in which only $r \le n$ of them are used without repetitions, requires making r selections. Therefore, for the first selection, there are n choices; for the second selection, there are (n – 1) choices; for the third, there are (n – 2) choices; and so on. Hence the events:

E_1 occurs in $m_1 = n$ ways

E_2 occurs in $m_2 = (n-1)$ ways

E_3 occurs in $m_3 = (n-2)$ ways

.

.

and E_r occurs in $m_r = (n-(r-1)) = (n-r+1)$ ways

Thus by the Fundamental Counting Principle

$$^nP_r = m_1 \cdot m_2 \cdot m_3 \cdots m_r = n(n-1)(n-2)\cdots(n-r+1)$$

(i) Since $^nP_r = n(n-1)(n-2)\cdots(n-r+1)$

$$^nP_r = n(n-1)(n-2)\cdots(n-r+1).\frac{(n-r)!}{(n-r)!}$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

(ii) Since $^nP_r = n(n-1)(n-2)\cdots(n-r+1)$

Now, putting r = n in the above, we obtain:

$$^nP_n = n(n-1)(n-2)\cdots(n-n+1) = n(n-1)(n-2)\cdots 1$$

$$= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$$

(iii) Since $^nP_n = n!$ then by using (ii), we obtain: $\frac{n!}{(n-n)!} = n!$

or $\frac{1}{0!} = 1 \Rightarrow 0! = 1$

Example 9: How many distinct six digit numbers can be formed from the integers 1, 2, 3, 4, .., 9 if each integer is used only once?

Solution: Since the total number of digits is 9 and each number we have to find, consists of six digits. No repetition is allowed. Therefore, this is a problem of permutation.

$\therefore$ The required number of six digit numbers $= {}^9P_6$

$$= \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3!}{3!} = \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3!}{3!}$$

$$= 9\cdot8\cdot7\cdot6\cdot5\cdot4 = 60480$$

Example 10: How many different words can be made out of the letters of the word "triangle"? How many of these will begin with t and end with e?

Solution:

(i) There are 8 different letters in the word "triangle". Therefore, the number of different words $= {}^8P_8 = \frac{8!}{(8-8)!} = \frac{8!}{0!} = 8! = 40320$

(ii) If 't' and 'e' occupy the first and last places, then we are left only with 6 different letters. Thus the number of different words in this case is ${}^6P_6 = 6! = 720$

Example 11: How many different arrangements of 10 objects taken 4 at a time can be made with one particular object (i) never occurs (ii) always?

Solution: There are 10 different objects and we are taking 4 at a time. Then the possible arrangements are ${}^{10}P_4 = \frac{10!}{(10-4)!} = 5040$ (1)

(i) Since one of the objects never occurs, so we are left with 9 objects. Thus the possible arrangements taking 4 at a time $= {}^9P_4 = \frac{9!}{(9-4)!} = 3024$ (2)

(ii) The possible arrangements that the particular object always occurs is obtained by subtracting (2) from (1), i.e. $5040 - 3024 = 2016$

6.2.4. Permutations with Repeated Elements

Consider the example of finding the number of different 9 digit numerals that can be formed from the digits: 6, 6, 6, 6, 5, 5, 5, 4, 3 and consider one such numeral: 665566543 (i)

With this ordering of the 9 digits, there are 4! Permutations of the digits 6 and 3! Permutations of the digits 5 which have no effect on the above numeral. Therefore, there are 4!·3! arrangements of digits in the numeral given in (i) which do not result in a distinguishable permutation of the given nine digits. Hence if X is the number of distinguishable permutations of the given 9 digits, then 4!·3!·X=9!, where 9! is the number of permutations of 9 distinct elements taken 9 at a time.

$$\therefore X = \frac{9!}{4!\cdot 3!} = 2520$$

The above example shows that in case of repeated elements, the number of permutations is reduced. Hence we have the following result.

Theorem: The number of distinguishable permutations of n elements taken all at a time, in which m_1 are alike, m_2 are alike, . . . and m_k are alike is

$$\frac{n!}{m_1!.m_2!...m_k!}$$

Proof: Let X be the required number of distinguishable permutations. Now, if we replace m_1 alike elements by m_1 different elements, then the number of permutations of m_1 distinct elements taken all at a time is $m_1!$. Similarly the replacement of $m_2, . . . ,m_k$ alike elements by different elements give rise to $m_2!, . . . m_k!$ permutations respectively.

Thus the simultaneous replacement of alike elements by different elements increases the number of permutations to X. $m_1!. m_2!. . . m_k!$

Since $n = m_1 + m_2 + . . . + m_k$, then the number of permutations of n distinct elements is n!

$$\therefore X.m_1!.m_2!...m_k! = n! \Rightarrow X = \frac{n!}{m_1!.m_2!...m_k!}.$$

Where X is generally denoted by, $\begin{pmatrix} n \\ m_1, m_2, ..., m_k \end{pmatrix}$

Thus $\begin{pmatrix} n \\ m_1, m_2, ..., m_k \end{pmatrix} = \frac{n!}{m_1!.m_2!...m_k!}$

Remember

We usually omit those digits, which occur once.

Example 12: Find the number of different arrangements that can be made out of the letters of the word "assassination" taken all together.

Solution: The total number of letters is 13, out of which 4 are s, 3 are a, 2 are i and 2 are n, so $n = 13$, $m_1 = 4$, $m_2 = 3$, $m_3 = 2$, $m_4 = 2$

Thus the required number of permutations $= \begin{pmatrix} n \\ m_1, m_2, m_3, m_4 \end{pmatrix} = \begin{pmatrix} 13 \\ 4,3,2,2 \end{pmatrix} = \frac{13!}{4!\cdot 3!\cdot 2!\cdot 2!}$

$= 10810800$

Example 13: How many eight – digit different numbers are possible using all of the digits 1, 1, 1, 1, 2, 2, 3, 4?

Solution: The total number of digit is 8, out of which four are 1s and two are 2s. So here $n = 8$, $m_1 = 4$, $m_2 = 2$,

thus the total eight digit number $= \binom{n}{m_1, m_2} = \binom{8}{4,2} = \frac{8!}{4!\cdot 2!} = 840$

6.2.5 Arrangements of Distinct Elements Round a circle

We have been arranging elements in a row and have seen that 4 elements can be arranged in a row in 4! = 24 different ways. Suppose we arrange these same 4 elements in a symmetric circular pattern. For example let us arrange A, B, C, D around a circle. One such arrangement is shown in Figure 6.1 and others in Figure 6.2.

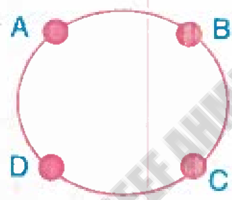


Figure 6.1.

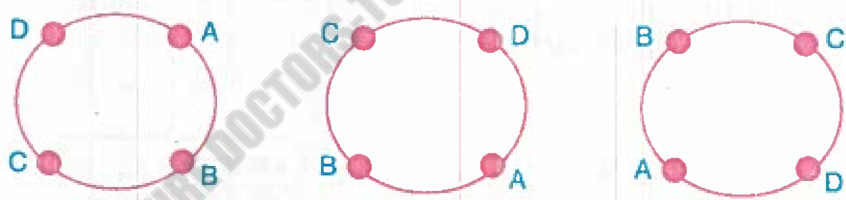


Figure 6.2

Now to check, whether these four arrangements are different or not. Let us ignore the positions of A, B, C, D and consider only their relative order as we go around the circle in a specific direction. We see that these four arrangements are the same. For example if we begin at A and move clockwise around any of the circles we get the same arrangement, ABCD and then back to A again. Thus the four different arrangements ABCD, BCDA, CDAB and DABC are not distinguishable in a circular arrangement.

In general, if there are X distinct circular arrangements of four elements, there would be $4\cdot X$ arrangements of these elements along a row. But since the number of arrangements along a row is 4! ,

then we have $4\cdot X = 4! \Rightarrow X = \frac{4!}{4} = \frac{4\cdot 3!}{4} = 3!$

Extending this, we have the following:

The number of distinguishable circular permutations of n elements is (n–1)!.

In arranging keys on a ring or different beads on a necklace, it is agreed that two arrangements are the same if one arrangement can be obtained from the other by turning over the ring (or the necklace) is reflection of one another. Thus in case of the Example 15 of four elements A, B, C, D, the following two arrangements are the same under such conditions (reflection of one another).

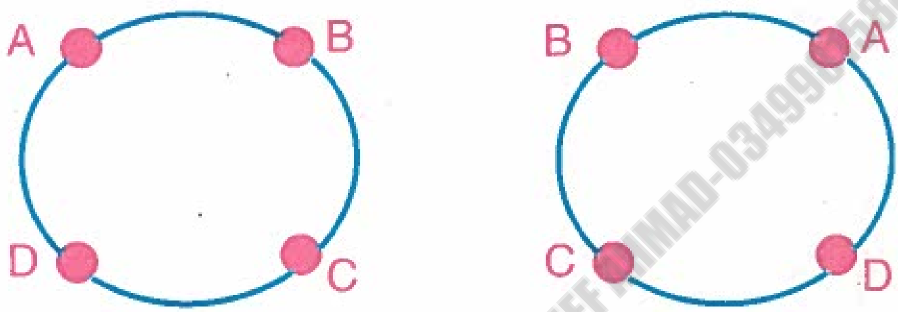


Figure 6.3

Consequently, there are three different arrangements of four different keys on a ring (or four different beads on a necklace), that is, the number of different arrangements is $\frac{(4-1)!}{2}=\frac{3!}{2}=3$

More generally, the number of different arrangements of keys on a ring (or n different beads on a necklace) is $\frac{(n-1)!}{2}$.

Example 14: In how many ways can six people be seated around a circular table?

Solution: In this case n = 6, so that six people can be seated around a circular table in (6 – 1)! = 5! = 120 ways.

Example 15: How many different necklaces can be formed by stringing eight beads of different colors?

Solution: The number of different necklaces is $\frac{(n-1)!}{2}$

So for n = 8 , we have $\frac{(8-1)!}{2}=\frac{7!}{2}=2520$ different necklaces.

EXERCISE 6.2

1. Evaluate (i) ${}^{6}P_{6}$ (ii) ${}^{20}P_{2}$ (iii) ${}^{16}P_{3}$
2. Solve for n (i) ${}^{n}P_{5} = 56({}^{n}P_{3})$ (ii) ${}^{n}P_{5} = 9({}^{n-1}P_{4})$ (iii) ${}^{n^2}P_{2} = 600$
3. Prove the following by Fundamental Principle of counting

   (i) ${}^{n}P_{r} = n({}^{n-1}P_{r-1})$ (ii) ${}^{n}P_{r} = {}^{n-1}P_{r} + r({}^{n-1}P_{r-1})$
4. In how many ways can a police department arrange eight suspects in a line up?
5. In how many ways can letters of the word 'Fasting' be arranged?
6. How many 4 digit numbers can be formed with the digits 2, 4, 5, 7, 9. (Repetitions not being allowed). How many of these are even?
7. How many three digit numbers can be formed from the digits 1, 2, 3, 4 and 5 if repetitions (i) are allowed (ii) are not allowed.
8. How many different arrangements can be formed of the word "equation" if all the vowels are to be kept together?
9. How many signals can be given by six flags of different colors when any number of them are used at a time?
10. In how many ways can five students be seated in a row of eight seats if a certain two students (i) insist on sitting next to each other?
    (ii) refuse to sit next to each other?
11. How many numbers each lying between 10 and 1000 can be formed with digits 2, 3, 4, 0, 8, 9 using only once?
12. How many different words can be formed from the letters of the following words if the letters are taken all at a time?
    (i) Bookworm (ii) Bookkeeper (iii) Abbottabad (iv) Letter
13. Find the number of permutations of the word 'EXCELLENCE'. How many of these permutations (i) begin with E (ii) begin with E and end with C (iii) begin with E and end with E (iv) do not begin with E. (v) contain two 2L's together (vi) do not contain 2L's together.
14. If five distinct keys are placed on a key ring, how many different orders are possible?
15. In how many ways can 7 people be arranged at a round table so that 2 particular persons always sit together?

6.3 Combination

So for, we have been concerned with permutations, which are ordered arrangements of elements of a set. Now, we focus our discussion on arrangements in which order is not important that is, subsets of a set.

6.3.1 Let S be a set containing n elements and suppose r is a positive integer such that $r \leq n$ .Then any subset of S containing r distinct elements is called a combination of n elements taken r at a time.

Notation: The notation, we use for the number of combinations of n elements taken r at a time is ${}^nC_r$ or $\binom{n}{r}$.

Example 16: Suppose $S = \{a, b, c, d\}$. Find the number of combinations by taking 3 letters at a time.

Solution: The subsets of S taken three elements at a time are:

$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$

Therefore, ${}^4C_3 = 4$

The distinction between permutations and combinations is that changing the order of a set of elements gives a different permutation but the same combination. For example in the above example there are four subsets of $\{a,b,c,d\}$, taken three at a time , because ${}^4C_3 = 4$. But the elements of each one of the four subsets can be arranged in a definite order in 3! or 6 different ways. Thus the total number of different arrangements in a definite order in all four subsets is

$$6 \cdot 4 = {}^4P_3 \quad \text{or} \quad 3! \cdot {}^4C_3 = {}^4P_3$$

or ${}^4C_3 = \dfrac{{}^4P_3}{3!} = \dfrac{4!}{3!(4-3)!}$ and we have the following important formula.

6.3.2 Theorem: Prove that ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ And hence deduce that

(i) $\binom{n}{n} = 1$, (ii) $\binom{n}{0} = 1$, (iii) $\binom{n}{1} = n$, (iv) $\binom{n}{n-1} = n$, (v) $\binom{n}{r} = \binom{n}{n-r}$

Proof: To find ${}^nC_r$, we must find the total number of subsets of r elements each of that can be obtained from a set of n elements. Since each of these combinations (subsets) contains r elements, which can be permuted among themselves in r! ways. Thus ${}^nC_r$ such combinations will give ${}^nC_r . r!$ permutations. But we know that the number of permutations of n elements taken r at a time is ^nP_r $\quad \therefore \quad {}^nC_r.r! = {}^np_r = \dfrac{n!}{(n-r)!} \Rightarrow {}^nC_r = \dfrac{n!}{r!(n-r)!}$

(i) If $r = n$, then $^nC_n = \dfrac{n!}{n!(n-n)!} = \dfrac{1}{0!} = 1,$ $\quad \because 0! = 1$

(ii) If $r = 0$, then $^nC_0 = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{n!} = 1$

(iii) If $r = 1$, then $^nC_1 = \dfrac{n!}{1!(n-1)!} = \dfrac{n(n-1)!}{(n-1)!} = n$

(iv) If $r = n-1$, then $^nC_{n-1} = \dfrac{n!}{(n-1)!(n-n+1)!} = \dfrac{n(n-1)!}{(n-1)!.1!} = n$

(v) Putting (n – r) for r, we have

$$^nC_{n-r} = \frac{n!}{(n-r)!\,(n-n+r)!} = \frac{n!}{(n-r)!\,r!} = {}^nC_r$$

Example 17: Prove that $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Solution: Taking L.H.S = $^nC_r + {}^nC_{r-1}$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$= \frac{(n+1)n!}{r(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{(n+1)!}{r!.(n-r+1)!} = \frac{(n+1)!}{r![(n+1)-r]!} = {}^{n+1}C_r = \text{R.H.S}$$

Did You Know ?

The number of combinations of n things r at a time is equal to the number of combinations of n things n – r at a time i.e. $^nC_r = {}^nC_{n-r}$

Such combinations are called complementary.

Put $r = n$, then $^nC_0 = {}^nC_n = 1$

Example 18: From 12 books in how many ways can a selection of 5 be made, (i) when one specified book is always included, (ii) when one specified book is always excluded?

Solution: (i) Since the specified book is to be included in every selection, we have only to choose 4 out of the remaining 11.

Hence the number of ways $= {}^{11}C_4$

$$= \frac{11\times10\times9\times8}{1\times2\times3\times4} = 330.$$

(ii) Since the specified book is always to be excluded, we have to choose the 5 books out of the remaining 11.

$$\text{Hence the number of ways} = {}^{11}C_5 = \frac{11\times10\times9\times8\times7}{1\times2\times3\times4\times5} = 462.$$

Example 19: Out of 14 men in how many ways can an eleven be chosen?

Solution: The required number $= {}^{14}C_{11} = {}^{14}C_3 = \dfrac{14\times13\times12}{1\times2\times3} = 364.$

EXERCISE 6.3

1. Solve the following for n.
   (i) ${}^{n}C_2 = 36$ (ii) ${}^{n+1}C_4 = 6.{}^{n-1}C_2$ (iii) ${}^{n^2}C_2 = 30.{}^{n}C_3$
2. Find n and r if ${}^{n}P_r = 840$ and ${}^{n}C_r = 35$
3. Find n when ${}^{2n}C_3 : {}^{n}C_2 = 36 : 3$
4. Prove that (i) ${}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^{n}C_r$ (ii) $r.{}^{n}C_r = n.{}^{n-1}C_{r-1}$
5. How many (i) straight lines (ii) triangles are determined by 12 points, no three of which lie on the same straight line.
6. Find the total number of diagonals of a hexagon.
7. Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?
8. A student is to answer 7 out of 10 questions in an examination. How many choices has he, if he must answer the first 3 questions?
9. An 8-person committee is to be formed from a group of 6 women and 7 men. In how many ways can the committee be chosen if (i) the committee must contain four men and four women? (ii) there must be at least two men? (iii) there must be at least two women? (iv) there must be more women than men?

6.4 Probability

Unconscious application of probability theory is very wide and indeed, practically every one is applying it without realizing. The phrases like "He is reliable", "He is a liar", "He is not likely to come" and so on are all probabilistic and we use them by "applying" probability theory. Basically, probability originated in problems related to games of chance and was developed mathematically by Pascal (1623 – 1662) and Fermat (1601 – 1665). Today, probability has grown far beyond the area of games of chance and has applications in genetics, insurance, physics, social sciences, engineering and medicine.

Before defining probability, we define and explain certain terms which are used in its definition.

6.4.1 (i) Statistical Experiment

Intuitively by an experiment one pictures a procedure being carried out under a certain set of conditions. The procedure can be repeated any numbers of times under the same set of conditions and upon completion of the procedure certain results are observed. The experiments are of two types

(a) Deterministic experiment An experiment is deterministic if, given the conditions under which the experiment is carried out, the outcome is completely determined. For example if pure water is brought to a temperature of 100^0 C and 760 mm Hg of atmospheric pressure the outcome is that the water will boil.

(b) Random experiment An experiment for which the outcome cannot be predicted except that it is known to be one of a set of possible outcomes, is called a random experiment.

For example (i) Tossing a coin (ii) Rolling a die.

Since our interest lies in the random experiment, so in this text by experiment we mean random experiment.

(ii) Sample space and an event

The set of all possible outcomes of a random experiment is called a sample space and is donated by S. The elements of S are called sample points or outcomes.

For example (a) Tossing a coin once, then

$S = \{H, T\}$ where H and T are the possible outcomes.

(b) Tossing a coin twice, then the possible outcomes in the sample space are

HH, HT, TH, TT.

(c) Rolling a pair of dice, then we have the following sample space

$S = \{ (i, j): i , j = 1, 2, 3, 4, 5, 6 \}$

$$= \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

Event: Let S be the sample space of an experiment. Any subset E of S is called an event associated with the experiment. For example E = {HH, TT} is an event associated with the experiment of tossing a coin twice.

(iii) Mutually Exclusive events

Two events are said to be mutually exclusive if they cannot both occur at the same time. Mathematically, it is expressed as:

If $A \cap B = \phi$, then A and B are mutually exclusive events.

For example rolling a die, let A be the event that even number has shown up while B be the event that odd number has shown up and C be the event that a number less than 4 has occurred.

Here S = {1, 2, 3, 4, 5, 6}

Let A = {even number has shown up } = {2, 4, 6}

B = {odd number has shown up } = {1, 3, 5}

and C = {a number less than 4 has occurred}= {1, 2, 3}

Now $A \cap B = \phi \Rightarrow$ A and B are mutually exclusive while $A \cap C = \{2\}$ and $B \cap C = \{1,3\}$ showing that A, C and B, C are not mutually exclusive.

(iv) Equally likely events

Two events are said to be equally likely if they have equal chances of happening. In other words, each event is as likely to occur as the other. For example rolling a die we have S = { 1, 2, 3, 4, 5, 6 } and each simple event $A_j = \{ j : j = 1, 2, 3, 4, 5, 6 \}$ is as likely to appear as the other. Hence they are equally likely events.

(v) Simple and compound events

Events of the form {s} are called simple events, while an event containing at least two sample points is called a compound event. For example E_1 = {HH} is a simple event and E_2 = { HH, TT } is a compound event associated with the experiment of tossing a coin twice.

If the random experiment results in s and $s \in A$, we say that the event A occurs or happens. The $\underset{j}{\cup} A_j$ occurs if at least one of the A_j occurs. The $\underset{j}{\cap} A_j$ occurs if all A_j occur.

If the event A occurs, then $\overline{A}$ (complement of A relative to S) fails to occur.

Did You Know

Favorable or Successful Outcomes

The outcomes which entail the happening of an event are said to be favorable (successful) to the event.
For example rolling a die, the number of outcomes favorable (successful) to the happening of event of even integers are three, i.e 2,4 and 6.

6.4.2 Let S be the sample space of a random experiment, and E be an event. The probability that an event E will occur, denoted by P (E) is given by

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{\text{the number of favorable (successful) outcome}}{\text{the total number of outcomes}}$$

$$= \frac{\text{no. of elements in the event E}}{\text{no. of elements in the sample space S}}$$

Since E is a subset of S, then obviously

$0 \le n(E) \le n(S)$ Dividing by n (S), we obtain

$$\frac{0}{n(S)} \le \frac{n(E)}{n(S)} \le \frac{n(E)}{n(S)} \text{ Or } 0 \le P(E) \le 1$$

Hence the probability of an event is always a number between 0 and 1 inclusive.

By the above definition, it is quite clear that $P(\phi) = 0$ and $P(S) = 1$ that is why ϕ is called an impossible event while S is called sure or certain event. If E and F are two events such that $P(E) < P(F)$, then we say that F is more likely to occur than E and if P (E) = P (F), the events E and F are equally likely.

Example 20: (a) If a coin is flipped, find the probability that a head will turn up.

(b) If a fair die is tossed, find the probability that an even number has shown up.

Solution: (a) Here S = { H , T }

Let A = { head has shown up } = { H }

Since, the outcomes are equally likely, then using the formula:

$$P(A)=\frac{n(A)}{n(S)}=\frac{1}{2}$$

(b) Here S = { 1, 2, 3, 4, 5, 6 }

Let B = { even number has shown up }= { 2, 4, 6 }

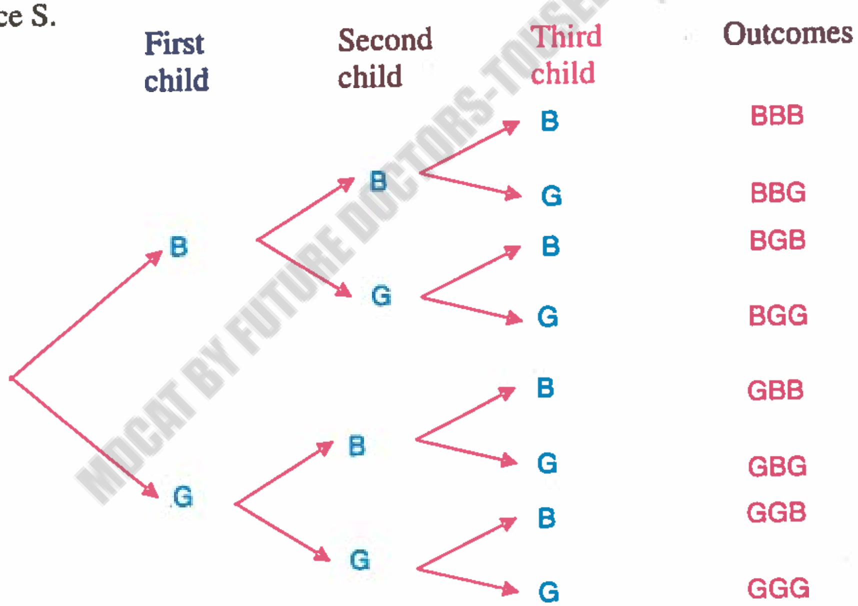
Since, the outcomes are equally likely, then we have $P(B)=\frac{n(B)}{n(S)}=\frac{3}{6}=\frac{1}{2}$

Example 21: In a three child family what is the probability of having

(i) three boys? (ii) at most one boy ?

(iii) at least one boy (iv) exactly one boy ?

Solution: Sometimes a tree diagram is very helpful in constructing a sample space S.



Hence S = { BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG } and the outcomes are equally likely.

(i) Let A = { having three boys } = { BBB } then $P(A)=\frac{n(A)}{n(S)}=\frac{1}{8}$

(ii) Let C = { having at most one boy } = { BGG, GBG, GGB, GGG }

then $P(C) = \frac{n(C)}{n(S)} = \frac{4}{8} = \frac{1}{2}$

(iii) Let D = { having at least one boy }
= { BBB, BBG, BGB, BGG, GBB, GBG, GGB }

then $P(D) = \frac{n(D)}{n(S)} = \frac{7}{8}$

(iv) Let E = { having exactly one boy }= { BGG, GBG, GGB }

then $P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$

EXERCISE 6.4

1. Let S = {1, 2, 3, 4, 5, 6} be the sample space of rolling a die. What is the probability of (i) Rolling a 5? (ii) Rolling a number less than one? (iii) Rolling a number greater than 0? (iv) Rolling a multiple of 3? (v) Rolling a number greater than or equal to 4?

2. A bag contains 4 white, 5 red and 6 green balls. 3 balls are drawn at random. What is the probability that (i) All are green (ii) All are white.

3. A true or false test contains eight questions. If a student guesses the answer for each question, find the probability:
   (i) 8 answers are correct. (ii) 7 answers are correct and 1 is incorrect.
   (iii) 6 answers are correct and 2 are incorrect.
   (iv) at least 6 answers are correct.

4. Three unbiased coins are tossed. What is the probability of obtaining
   (i) all heads (ii) two heads (iii) one head
   (iv) at least one head (v) at least two heads (vi) All tails.

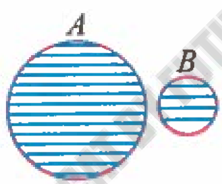
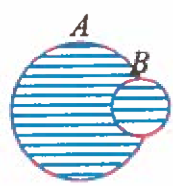
5. A committee of 5 person is to be selected at random from 6 men and 4 women. Find the probability that the committee will consist of
   (i) 3 men and 2 women (ii) 2 men and 3 women.

6. If one card is drawn at random from a well shuffled pack of 52 cards. Then find the probability of each of the following.
   (i) Drawing an ace card, (ii) Drawing either spade or hearts,
   (iii) Drawing a diamond card, (iv) Drawing a face card,
   (v) Not drawing an ace of hearts.
7. Two dice are thrown simultaneously. Find the probability of getting:
   (i) doublet of even numbers (ii) a sum less than 6 (iii) a sum more than 7
   (iv) a sum greater than 10 (v) a sum at least 10 (vi) six as the product
   (vii) an even number as the sum (viii) an odd number as the sum
   (ix) a multiple of 3 as the sum (x) sum as a prime number

6.4.3 Laws of Probability

It is easier to compute the probability of an event from known probabilities of other events. This is true if the event can be expressed as the union or intersection of two other events or as the complement of an event. Some basic elementary laws of probability are given below in the form of theorems.

6.4.4 Use Venn diagrams to find the probability for the occurrence of an event

If A and B are disjoint	If A and B are overlapping	If $B \subset A$
		

We know that if A and B are two sets, then the shaded parts in the following diagram represent $A \cup B$.
The above diagrams help us in understanding the formulae about the sum of two probabilities.
We know that:
$P(E)$ is the probability of the occurrence of an event E.
If A and B are two events, then
$P(A)$ = the probability of the occurrence of event A;

$P(B)$ = the probability of the occurrence of event B;

$P(A \cup B)$ = the probability of the occurrence of $A \cup B$;

$P(A \cap B)$ = the probability of the occurrence of $A \cap B$;

The formulae for the addition of probabilities are:

i) $P(A \cup B) = P(A) + P(B)$, when A and B are disjoint.

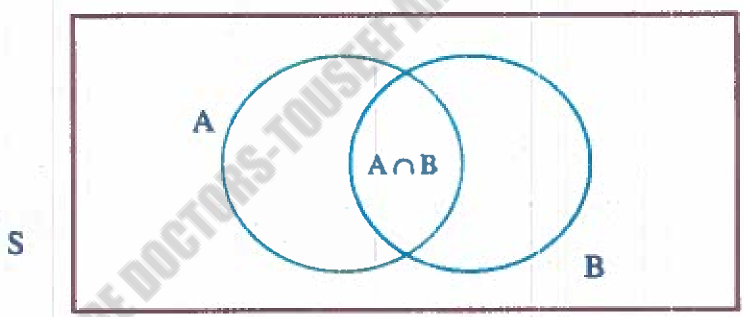
ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

when A and B are overlapping or $B \subseteq A$.

Theorem: If A and B are any two events in a sample space S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof: From the Venn diagram, it is clear that



$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

and $n(A \cap B)$ has been subtracted simply because it has been considered twice.

Now, by definition we have

$$P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{n(A) + n(B) - n(A \cap B)}{n(S)}$$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

$$= P(A) + P(B) - P(A \cap B)$$

This law is generally called, addition law of probability.

Corollary 1: If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

Proof: Since A and B are mutually exclusive events, then

$$A \cap B = \varphi \quad \text{and} \quad P(A \cap B) = P(\varphi) = 0$$

Hence $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ reduces to

$$P(A \cup B) = P(A) + P(B)$$

Now, generalizing the above, we have the following:

Corollary 2: If $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then

$$P\left(A_1 \cup A_2 \cup \cdots \cup A_n\right) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

Example 22: One integer is chosen at random from the numbers 1, 2, 3, . . . , 50. What is the probability that the chosen number is divisible by 6 or 8 ?

Solution: Here $S = \{1, 2, 3, \cdots, 50\}$ and $n(S) = 50$

Let A = { number is divisible by 6 }= { 6, 12, 18, 24, 30, 36, 42, 48 }

and B = { number is divisible by 8 }

= { 8, 16, 24, 32, 40 , 48 } then $A \cap B = \{24, 48\}$

Now, substituting $P(A) = \frac{8}{50}$, $P(B) = \frac{6}{50}$ *and* $P(A \cap B) = \frac{2}{50}$

in the following, we obtain

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{8}{50} + \frac{6}{50} - \frac{2}{50} = \frac{12}{50} = \frac{6}{25}$$

Example 23: If two dice are rolled, find the probability of obtaining a total of 7 or 11.

Solution: Here $S = \{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}$ and $n(S) = 36$

Let A = { a total of 7 occurs }

= { (6,1), (5,2), (4,3), (3, 4), (2,5), (1,6) }

and B = { a total of 11 occurs }

$= \{ (6,5), (5,6) \}$ then $A \cap B = \varphi$

Now $P(A) = \dfrac{6}{36}$ and $P(B) = \dfrac{2}{36}$

Since A and B are mutually exclusive, so we have

$$P(A \cup B) = P(A) + P(B) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$$

Complementary events

Suppose we divide a sample space S into two subsets (events) E and E′ such that

(i) $E \cap E' = \phi$ and (ii) $E \cup E' = S$

Then E' is called the complement of E relative to S and E and E' are called complementary events.

Theorem: If E and E' are complementary events, then $P(E') = 1 - P(E)$

Proof: Since $E \cup E' = S$ Then $P(E \cup E') = P(S)$

or $P(E) + P(E') = 1, \quad \because E \cap E' = \varphi$

or $P(E') = 1 - P(E)$

Example 24: A coin is tossed 6 times in succession. What is the probability that at least one head occurs?

Solution: Tossing a coin 6 times in succession, we have $n(S) = 2^6 = 64$

Let $E = \{$ at least 1 H occurs $\}$ then $E' = \{$ no H occurs $\}$

and $P(E') = \dfrac{1}{64}, \quad \because$ there is only one outcome event, where all tails occur.

$\therefore \quad P(E) = 1 - P(E') = 1 - \dfrac{1}{64} = \dfrac{63}{64}$

6.4.5 Conditional Probability

The probability of an event may change if the information of the occurrence of another event is given. For example, if an adult is selected at random from certain population, the probability of that person having lung cancer would not be too high. However, if information that the person is also a heavy smoker is provided, then one would certainly want to revise the probability upward.

Let A = { An adult has lung cancer }

and B = { An adult is a heavy smoker }

Then the probability of an event A, given the occurrence of another event B, is called a conditional probability and is denoted by $P(A|B)$.

For events A and B in an arbitrary sample space S, we define the conditional probability of A given B by $P(A|B)=\dfrac{P(A\cap B)}{P(B)},\quad P(B)>0.$

Similarly, $P(B|A)=\dfrac{P(A\cap B)}{P(A)},\quad P(A)>0.$

Example 25: What is the probability of rolling a prime number in tossing a die, given that an odd number has turned up?

Solution: Here S = { 1, 2, 3, 4, 5, 6 }

Let A= { a prime number has rolled }= { 2, 3, 5 }

and B= { an odd number has turned up } = { 1, 3, 5 }

then $A\cap B=\{3,5\}$

We have $P(B)=\dfrac{3}{6}$ and $P(A\cap B)=\dfrac{2}{6}$

Now, using the formula $P(A|B)=\dfrac{P(A\cap B)}{P(B)},\quad P(B)\neq 0$

$$P(A|B)=\dfrac{\frac{2}{6}}{\frac{3}{6}}=\dfrac{2}{3}\quad \text{Since}\quad P(B|A)=\dfrac{P(A\cap B)}{P(A)}$$

or $\quad P(A\cap B)=P(A)P(B|A)$

This shows that the conditional probability can be used in expressing the probability of the intersection of a finite number of events and we have the following theorem known as **the multiplicative theorem**.

> If A and B are any two events in a sample space S then
>
> $$P(A\cap B)=P(A)P(B|A),\quad P(A)\neq 0$$
>
> $$=P(B)P(A|B),\quad P(B)\neq 0$$
>
> The above theorem can be easily extended to a finite number of events.
> For example in case of three events A, B and C it becomes:
>
> $$P(A\cap B\cap C)=P(A)P(B|A)P(C|A\cap B)$$

Example 26: An urn contains three red and seven green balls. A ball is drawn, not replaced and another is drawn. Find the following.

(i) P (red and red) (ii) P (red and green).

Solution: Total number of balls = 10

(i) Let A = { the 1st ball drawn is red }
and B = { the 2nd ball drawn is red }

So using the multiplicative theorem,

$$P(\text{red and red})=P(A\cap B)=P(A)P(B|A)$$

Substituting $P(A)=\dfrac{3}{10}$ and $P(B|A)=\dfrac{2}{9}$

We obtain, $P(\text{red and red})=\dfrac{3}{10}\cdot\dfrac{2}{9}=\dfrac{1}{15}$

(ii) Let C = { the 1st ball drawn is red }
and D = { the 2nd ball drawn is green }

So using the multiplicative theorem again.

$$P(\text{red and green})=P(C\cap D)=P(C)P(D|C)$$

Substituting $P(C)=\frac{3}{10}$ and $P(D|C)=\frac{7}{9}$

We obtain, $P(\textit{red and green})=\frac{3}{10}\cdot\frac{7}{9}=\frac{7}{30}$

6.4.6 Dependent and Independent Events

In general $P(A|B)$ and $P(A)$ are not equal. However, there is an important class of events for which they are, If $P(A|B)=P(A)$, then the knowledge of B occurring does not change the probability of A and we say that A is independent of B. Similarly, if $P(B|A)=P(B)$, we say that B is independent of A. Thus two events A and B are said to be independent if the occurrence (or non-occurrence) of one does not affect the probability of the occurrence (and hence non-occurrence) of the other, otherwise they are called dependent.

Illustration1: In the simultaneous throw of two coins, 'getting a head' on first coin and 'getting a tail on the second coin are independent events.

Illustration2: When a card is drawn from a pack of well shuffled cards and replaced before the second card is drawn, the result of second draw is independent of first draw.

The following theorem gives the probabilities of simultaneous occurrence of two independent events.

Theorem: If A and B are independent events, then $P(A\cap B)=P(A)P(B)$.

Proof: Since multiplicative theorem gives that:

$$P(A\cap B)=P(A)P(B|A) \qquad \text{(i)}$$

$$=P(B)P(A|B) \qquad \text{(ii)}$$

Further, A and B are independent, then we have $P(B|A)=P(B)$ and $P(A|B)=P(A)$ substituting in (i) and (ii) we get the required result:

$$P(A \cap B) = P(A)P(B)$$

The above theorem can be extended to any finite number of mutually independent events. If A_1 , A_2 , A_3 , ... , A_n are mutually independent events, then

$$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) = P(A_1)P(A_2)\,P(A_3)\cdots P(A_n)$$

Example 27: A space shuttle has four independent computer control systems. If the probability of failure of any one system is 0.001, what is the probability of failure of all four systems ?

Solution: Let $E_i = \{$ failure of system i , $i = 1, 2, 3, 4$ $\}$

Since the events E_i , $i = ,1\ 2, 3, 4$ are given to be independent, so using the following.

$$P(E_1 \cap E_2 \cap E_3 \cap E_4) = P(E_1)P(E_2)P(E_3)P(E_4), \quad i = 1,2,3,4$$

$$= (0.001)^4 = 0.000000000001$$

EXERCISE 6.5

1. Suppose events A and B are such that $P(A) = \frac{2}{5}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{1}{2}$. Find $P(A \cap B)$.
2. If A and B are 2 events in a sample space S such that $P(A) = \frac{1}{2}, P(\overline{B}) = \frac{5}{8}, P(A \cup B) = \frac{3}{4}$. Find (i) $P(A \cap B)$ (ii) $P(\overline{A} \cap \overline{B})$
3. Given $P(A) = 0.5$ and $P(A \cup B) = 0.6$, find $P(B)$ if A and B are mutually exclusive.
4. A bag contains 30 tickets numbered from 1 to 30. One ticket is selected at random. Find the probability that its number is either odd or the square of an integer.
5. A student finds that the probability of passing an algebra test is $\frac{8}{9}$. What is the probability of failing the test?
6. In the two dice experiment, given that the first die shows 4, what is the probability that the second die shows a number greater than 4 ?
7. One card is drawn from a pack of 52 cards, what is the probability that the card drawn is neither red nor king.

8. If a pair of dice is thrown, find the probability that the sum of digits is neither 7 nor 11.

9. Ajmal and Bushra appear in an interview for 2 vacancies. The probability of their selection being $\frac{1}{7}$ and $\frac{1}{5}$ respectively. Find the probability that

(i) both will be selected (ii) only one is selected

(iii) none will be selected (iv) at least one of them will be selected.

10. A basket contains 20 apples and 10 oranges out of which 5 apples and 3 oranges are defective. If a person takes out 2 at random what is the probability that either both are apples or both are good?

REVIEW EXERCISE 6

1. **Choose the correct option**

(i) In how many ways can we name the vertices of a pentagon using any five of the letters O, P, Q, R, S, T, U in any order?

(a) 2520 (b) 9040 (c) 5140 (d) 4880

(ii) How many two-digit odd numbers can be formed from the digits {1,2, 3, 4, 5, 6, 7} if repeated digits are allowed?

(a) 14 (b) 42 (c) 28 (d) 21

(iii) How many six-digit numbers can be formed from the digits {2, 3, 4, 6, 7, 8} without repetition if the digits 3 and 7 must be together?

(a) 120 (b) 180 (c) 144 (d) 96

(iv) Evaluate $\frac{(n+2)!(n-2)!}{(n+1)!(n-1)!}$

(a) $(n-3)$ (b) $(n-1)$ (c) $\frac{(n+1)}{(n+2)}$ (d) $\frac{(n+2)}{(n-1)}$

(v) In how many different ways can 5 couples be seated around a circular table if the couples must not be separated?

(a) 768 (b) 724 (c) 844 (d) 696

(vi) A committee of 4 people will be selected from 8 girls and 12 boys in a class. How many different selections are possible if at least one boy must be selected?

(a) 2865 (b) 3755 (c) 4225 (d) 4775

(vii) The number of all possible matrices of order 3x3 with each entry 0 and 1 is:

(a) 18 (b) 27 (c) 512 (d) 81

(viii) How many diagonals can be drawn in a plane figure of 8 sides?

(a) 21 (b) 20 (c) 35 (d) 81

(ix) If $P(A)=\frac{1}{2}, P(B)=0$, then $P(A|B)$ is

(a) 0 (b) $\frac{1}{2}$ (c) not defined (d) 1

(x) If A and B are events such that $P(A|B)=P(B|A)$ then

(a) $A\subset B$ but $A\neq B$ (b) $A=B$ (c) $A\cap B=\varphi$ (d) $P(A)=P(B)$

2. (i) If ${}^{2n}C_r={}^{2n}C_{r+2}$; find r. (ii) If ${}^{18}C_r={}^{18}C_{r+2}$; find ${}^{r}C_5$.

3. ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800:1$, find r.

4. In how many distinct ways can $x^4y^3z^5$ be expressed without exponents?

5. In how many different ways can be six children seated at a round table if a certain two children (i) refuse to sit next to each other? (ii) insist on sitting next to each other?

6. Six people including Faisal and Saima are to be seated around a circular table. Find the probability that Faisal and Saima are seated next to each other.

7. If $P(A)=0.8, P(B)=0.5, P(B|A)=0.4$,

find (i) $P(A\cap B)$ (ii) $P(A|B)$ (iii) $P(A\cup B)$.

8. How many 6-digited telephone numbers can be constructed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, if each number starts with 35 and no digits appears more than once.

9. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?

10. A party of n men is to be seated round a circular table. Find the probability that two particular men sit together.

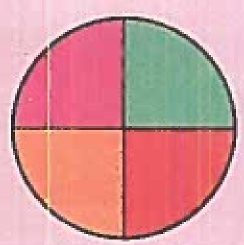
11. Given the following spinner, determine the probability:

P (Orange)

P (Red or Green)

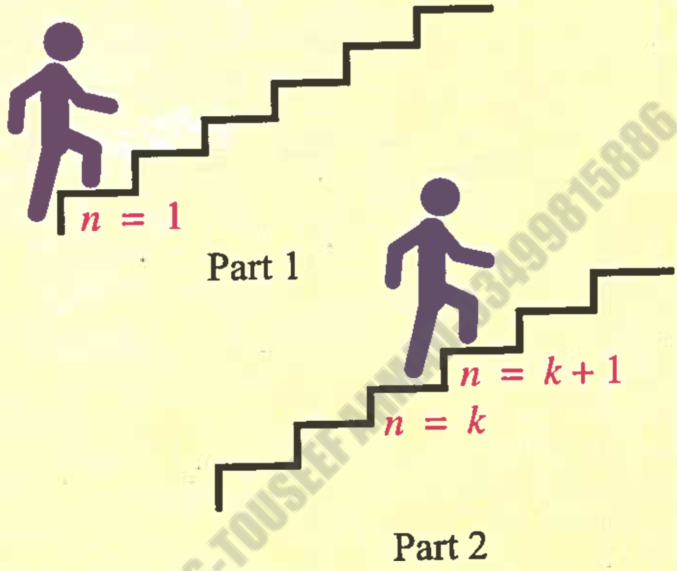
P (Not Red)

P (Pink)



UNIT 7

MATHEMATICAL INDUCTION AND BINOMIAL THEOREM



STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Describe the principle of mathematical induction.
- Apply the principle to prove the statements, identities or formulae.
- Use Pascal's triangle to find the expansion of $(x + y)^n$ where n is a small positive integer.
- State and prove binomial theorem for positive integral index
- Expand $(x + y)^n$ using binomial theorem and find its general term.
- Find the specified term in the expansion of $(x + y)^n$
- Expand $(1 + x)^n$ where n is a positive integer and extend this result for all rational values of n.
- Expand $(1 + x)^n$ in ascending powers of x and explain its validity/convergence for $|x| < 1$ where n is a rational number.
- Determine the approximate values of the binomial expansions having indices as -ve integers or fractions.

7.1. Introduction

To understand the basic principles of mathematical induction, suppose a set of thin rectangular tiles are placed as shown in the following Figure (7.1).

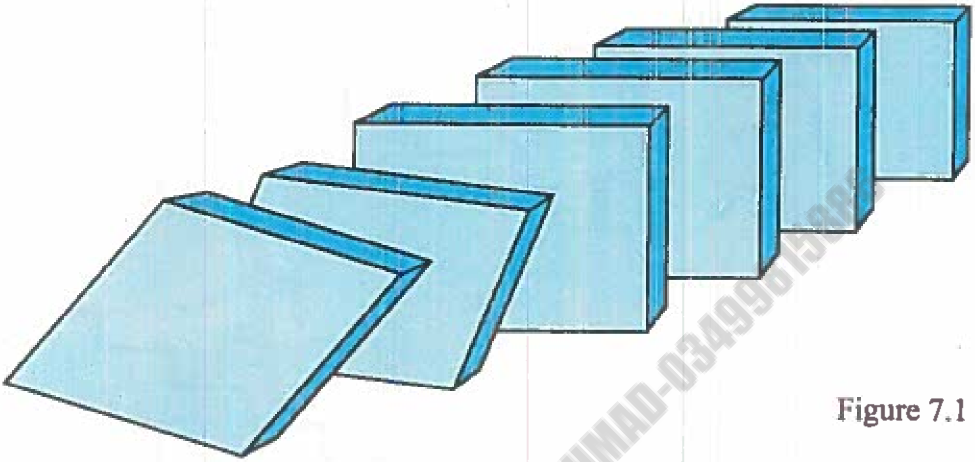


Figure 7.1

When the first tile is pushed in the indicated direction, all the tiles will fall. To be absolutely sure that all the tiles will fall, it is sufficient to know that

(a) The first tile falls, and

(b) In the event that any tile falls its successor necessarily falls.

This is the underlying principle of mathematical induction.

We know, the set of natural numbers **N** is a special ordered subset of the real numbers. In fact, **N** is the smallest subset of **R** with the following property.

A set **S** is said to be an inductive set if $1 \in \mathbf{S}$ and $x+1 \in \mathbf{S}$ whenever $x \in \mathbf{S}$.

Since **N** is the smallest subset of **R** which is an inductive set, it follows that any subset of **R** that is an inductive set must contain **N**.

Mathematical induction is one of the developed techniques of proof in the history of mathematics. It is used to check conjectures about the outcomes of processes that occur repeatedly and according to definite patterns.

For example:

$$1 + 3 + 5 + \dots\dots + (2n-1) = n^2 \qquad (1)$$

$$1 + 2 + 3 + \dots\dots + n = \frac{n(n+1)}{2} \qquad (2)$$

$$1^2 + 2^2 + 3^2 + \dots\dots + n^2 = \frac{n(n+1)(2n+1)}{2} \qquad (3)$$

are all propositions, statements which involve the natural number n. Equation (1) above asserts that the sum of first n positive odd integers is equal to the square of n. We see that the L.H.S. of (1) reduces simply to:

$$1 = 1 \quad \text{if} \quad n = 1$$

$$1 + 3 = 4 = 2^2 \quad \text{if} \quad n = 2$$

$$1 + 3 + 5 = 9 = 3^2 \quad \text{if} \quad n = 3 \text{ and so on.}$$

It is impossible to verify (1) for each $n \in N$, because it involves infinitely many calculations which never end. To avoid such situations, the principle of mathematical induction is applied.

7.1.1. The Principle of Mathematical Induction

The principle of mathematical induction is stated as follows.

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer.

Suppose the following two statements are true.

1. P(a) is true.
2. For all integers $k \geq a$, if P(k) is true then P(k+1) is true.

   Then the statement for all integers $n \geq a$; P(n) is true.

The principle of mathematical induction is explained through the following examples.

Example 1: Prove that for every $n \in N$, $1 + 2 + 3 + \ldots\ldots + n = \dfrac{n(n+1)}{2}$

Solution: **Step 1.** For n=1, the statement becomes

$$1 = \frac{1(1+1)}{2}. \quad \leftarrow \textit{basis } (p(1))$$

Thus the statement is true for n =1

Step 2. Let us assume that the statement be true for $n = k \in N$, that is, we assume

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}. \quad \leftarrow \text{inductive hypothesis } (P(k))$$

Step 3. Let n = k+1 and consider

$$(1 + 2 + \cdots + k) + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

(adding k+1 to both sides of P(k))

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+2)(k+1)}{2} \qquad = \frac{(k+1)(k+2)}{2}$$

Which is just the form taken by the proposition when n = k+1. So the above proposition is true for n=k+1 and thus by the principle of mathematical induction, it is true for all positive integers n.

Example 2: **(i) Find 2+4+6+......+500**

(ii) Find 5+6+7+8+......+50

(iii) Find an integer h≥2, find 1+2+3+.....+(h−1)

Solution:

(i) $2+4+6+\cdots+500 = 2\cdot(1+2+3+\cdots+250)$

$$= 2\cdot\left(\frac{250\cdot 251}{2}\right)$$ (by applying the formula for the sum of the first n with n =250)

$$= 62{,}750$$

(ii) $5+6+7+8+\cdots+50 = (1+2+3+\cdots+50) - (1+2+3+4)$ (by applying the formula for the sum of the first n with n =50)

$$= \frac{50\cdot 251}{2} - 10$$

$$= 1265$$

(by applying the formula for the sum of the first n with n =h− 1)

(iii) $1+2+3+\cdots+(h-1) = \dfrac{(h-1).[(h-1)+1]}{2} = \dfrac{(h-1).h}{2}$

Example 3: Prove that $1^2+2^2+3^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Solution: **Step 1.** For n = 1, the proposition becomes

$$1^2 = 1 = \frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = 1.$$ Thus it is true for n =1

Step 2. Suppose the proposition is true for n = k, then

$$1^2+2^2+3^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6} \qquad \text{(i)}$$

Step 3. let $n = k + 1$ and consider

$$1^2+2^2+3^2+\cdots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \quad \text{(Adding } (k+1)^2 \text{ to both sides of (i))}$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{(k+1)\{k(2k+1)+6(k+1)\}}{6}$$

$$= \frac{(k+1)\{2k^2+k+6k+6)\}}{6} = \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Which is just the form taken by the proposition for $n = k + 1$. So the above proposition is true for $n = k+1$ and hence by the principle of mathematical induction, it is true for all positive integer n.

It must be noted that the application of the principle of mathematical induction is not limited only to P(n) stated by means of an equation. The principle can also be applied in cases where no equation is involved as we shall see in the following examples.

Example 4: Show that a–b is a factor of $a^n - b^n$ for all positive integer n.

Solution: To show that $a - b$ is a factor of $a^n - b^n$, we will use induction on n.

Step 1. Let $n = 1$, then $a^n - b^n = a - b$ and since $a - b$ divides $a - b$, so $a - b$ is a factor of $a - b$. Therefore the above statement is true for $n = 1$.

Step 2. Let the above statement is true for $n = k$ then $a-b$ is a factor of $a^k - b^k$.

$\Rightarrow$ $a - b$ divides $a^k - b^k$ and as such we can write

$a^k - b^k = (a-b)\,Q$ ....(1) where Q is the quotient.

Step 3. Let $n = k + 1$ and consider $a^{k+1} - b^{k+1}$. We can write

$$a^{k+1} - b^{k+1} = a^k a - b^k b \quad \text{(Adding and subtracting the term } ab^k\text{)}$$

$$= a^k a - ab^k + ab^k - b^k b$$

$$= a(a^k - b^k) + b^k(a-b) = a(a-b)\,Q + b^k(a-b) \quad \text{(Using 1)}$$

$$= (a-b)\,[\,aQ + b^k\,]$$

$\Rightarrow$ $a - b$ divides $a^{k+1} - b^{k+1}$ with quotient $aQ + b^k$

$\Rightarrow$ $a - b$ is a factor of $a^{k+1} - b^{k+1}$

Therefore the above statement is true for n = k + 1 and hence by the principle of induction it is true for all positive integer n.

Example 5: Prove that if n is a positive odd integer then $x + y$ is a factor of $x^n + y^n$

Solution: Since n is given to be a positive odd integer, so we can write $n=2m-1$ where m is a positive integer. Therefore $x^n+y^n=x^{2m-1}+y^{2m-1}$

To prove the above statement, we will use the method of induction on m.

Step 1. Let m = 1, then $x^{2m-1} + y^{2m-1} = x^{2\cdot 1-1} + y^{2\cdot 1-1} = x + y$ and since $x + y$ divides $x + y$, so $x + y$ is a factor of $x + y$. Therefore the above statement is true for m = 1.

Step 2. Let the above statement is true for m = k then $x + y$ is a factor of $x^{2k-1} + y^{2k-1}$

$\Rightarrow$ $x + y$ divides $x^{2k-1} + y^{2k-1}$.

So we can write $x^{2k-1} + y^{2k-1} = (x + y)\,Q$ (1) where Q is the quotient.

Step 3. Now let m = k + 1 and consider

$$x^{2(k+1)-1} + y^{2(k+1)-1} = x^{2k+2-1} + y^{2k+2-1} = x^{2k-1+2} + y^{2k-1+2} = x^{2k-1}.\,x^2 + y^{2k-1}.\,y^2$$

$$= x^{2k-1}x^2 + y^{2k-1}x^2 - y^{2k-1}.\,x^2 + y^{2k-1}y^2$$

$$= x^2\,[x^{2k-1} + y^{2k-1}] + y^{2k-1}(y^2 - x^2)$$

$$= x^2\,(x+y)Q + y^{2k-1}(y-x)(y+x) \qquad \text{(Using 1)}$$

$$= x^2\,(x+y)Q + y^{2k-1}\,(y-x)(x+y) = (x+y)\,[x^2\,Q + y^{2k-1}(y-x)]$$

or $x^{2(k+1)-1} + y^{2(k+1)-1} = (x+y)\,Q_1$ where $Q_1 = x^2\,Q + y^{2k-1}(y-x)$

$\Rightarrow x+y$ is a factor of $x^{2(k+1)-1} - y^{2(k+1)-1}$

So the above statement is true for $m = k+1$ and hence by induction it is true for all positive integral values of m.

Therefore $x+y$ is factor of $x^n + y^n$ where n is a positive odd integer.

7.1.2 General (extended) form of principle of Mathematical Induction

Sometimes it happens that a given statement and proposition does not hold for first few positive integral values of n but after those values of n it becomes true; For example let us consider the statement $n^2 > n+3$

We see that when n=1 then $1 > 1+3$ or $1 > 4$ which is false.

When $n = 2$ then $2^2 > 2+3$ and $4 > 5$ which is again false.

When $n = 3$ then $3^2 > 3 + 3$ or $9 > 6$ which is true. That is, the above statement is false for $n = 1$ and 2 but is true for all values of n greater than 2.

Similarly if we consider the statement $n^3 > 4n^2 + n + 1$ then this statement is not true for $n = 1,2,3,4$ but it becomes true forn = 5 and higher values.

In such situations the principle of mathematical induction is defined as under:

Let P(n) is a given statement or proposition such that.

(i) P(n) is true for n = m, where m is the least positive integer.

(ii) If P(n) is true for $n = k$ where $k > m$ then p(n) is also true for $n = k+1$

We then say that P(n) is true for all integral values of $n \geq m$.

This is called general (extended) form of the principle of mathematical induction.

Example 6: Prove that $n^3 > 4n^2 + n + 1$ for $n \geq 5$

We are to prove that $n^3 > 4n^2 + n + 1$ for $n \geq 5$

Solution: In this case our induction will start from n=5

Step 1. Let $n = 5$, then $n^3 = 5^3 = 125$ and

$4n^2 + n + 1 = 4(5)^2 + 5 + 1 = 100 + 5 + 1 = 106$

Clearly $125 > 106$ so the above statement is true for $n = 5$

Step 2. Let us assume that the above statement is true for $n = k \geq 5$ then,

$k^3 > 4k^2 + k + 1$ (1)

Step 3. Now let $n = k+1$, then $n^3 = (k+1)^3$

and so $(k+1)^3 = k^3 + 3k^2 + 3k + 1 > 4k^2 + k + 1 + 3k^2 + 3k + 1$

$\Rightarrow (k+1)^3 > 4k^2 + 3k^2 + 4k + 2$ (using (1))

$\Rightarrow (k+1)^3 > 4k^2 + 3k \cdot k + 4k + 2 \Rightarrow (k+1)^3 > 4k^2 + 3k \cdot 5 + 4k + 2$ as $k \geq 5$

$\Rightarrow (k+1)^3 > 4k^2 + 15k + 4k + 2$ $\Rightarrow (k+1)^3 > 4k^2 + 9k + 6k + 4k + 2$

$\Rightarrow (k+1)^3 > 4k^2 + 9k + 10k + 2$ $\Rightarrow (k+1)^3 > 4k^2 + 9k + 6k + 4k + 2$

$\Rightarrow (k+1)^3 > 4k^2 + 9k + 6$ (as $6k + 4k + 2 > 6$)

$\Rightarrow (k+1)^3 > 4k^2 + 8k + k + 4 + 2$ $\Rightarrow (k+1)^3 > 4k^2 + 8k + 4 + k + 2$

$\Rightarrow (k+1)^3 > 4(k^2 + 2k + 1) + k + 1 + 1$ $\Rightarrow (k+1)^3 > 4(k+1)^2 + (k+1) + 1$

Which is of the form (1) for $n = k + 1$, so the given proposition is true for n=k+1, thus by induction it is true for all $n \geq 5$.

Example 7: Prove that $2^n < \binom{2n}{n}$ for n >1

Solution: We are to prove that $2^n < \binom{2n}{n}$ for n >1.......... (1)

Step 1.
Let n = 2 , then $2^n = 2^2 = 4$ and $\binom{2n}{n} = \binom{2\cdot 2}{2} = \binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4\cdot 3\cdot 2}{2\cdot 2} = \frac{24}{4} = 6$

Therefore 4<6 and once $2^n < \binom{2n}{n}$ is true for n = 2

Step 2. Let us suppose that the above assertion is true for n=k for k>1, then

$$2^k < \binom{2k}{k} \quad \text{or} \quad 2^k < \frac{2k!}{k!(2k-k)!}$$

$$2^k < \frac{2k!}{k!\,k!} \ldots\ldots\ldots\ldots \quad (2)$$

Step 3. Let n = k+1 and consider 2^{k+1}, we can write

$$2^{k+1} = 2^k \cdot 2 < \frac{2k!\cdot 2}{k!\,k!} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Now $\frac{(2k+2)(2k+1)}{(k+1)^2} = \frac{2(k+1)(k+k+1)}{(k+1)^2} = \frac{2(k+k+1)}{k+1} = 2\left[\frac{k}{k+1} + \frac{k+1}{k+1}\right]$

$$= 2\left[\frac{k}{k+1} + 1\right] = \frac{2k}{k+1} + 2 > 2 \text{ as } k > 1$$

$\Rightarrow 2 < \frac{(2k+2)(2k+1)}{(k+1)^2}$ From (3), we have

$2^{k+1} < \frac{2k!\cdot 2}{k!\cdot k!} < \frac{2k!}{k!\cdot k!}\cdot\frac{(2k+2)(2k+1)}{(k+1)^2}$ or, $2^{k+1} < \frac{(2k+2)(2k+1)\,2k!}{k!(k+1)\,k!(k+1)}$

$2^{k+1} < \frac{(2k+2)!}{k!(k+1)\,k!(k+1)} \Rightarrow 2^{k+1} < \frac{(2k+2)!}{(k+1)!\,(k+1)!}$

$2^{k+1} < \binom{2k+2}{k+1}$ which is of the form (1) when n is replaced by k+1.

So the given statement is true for n =k+1 and hence it is true for all n > 1.

Thus $2^n < \binom{2n}{n}$ for n > 1.

EXERCISE 7.1

Establish the formulas given below by mathematical induction.

1. $2+4+6+\cdots+2n=n(n+1)$
2. $1+5+9+\ldots\ldots+(4n-3) = n(2n-1)$
3. $3+6+9+\ldots\ldots+3n = \dfrac{3n(n+1)}{2}$
4. $3+7+11+\cdots+(4n-1)=n(2n+1)$
5. $1^3+2^3+3^3+\cdots+n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$
6. $1(1!)+2(2!)+3(3!)+\cdots+n(n!) = (n+1)!-1$
7. $1\cdot2+2\cdot3+3\cdot4+\ldots\ldots+n(n+1) = \dfrac{n(n+1)(n+2)}{3}$
8. $1+2+2^2+2^3+\ldots\ldots+2^{n-1}=2^n-1$
9. $\dfrac{1}{3}+\dfrac{1}{9}+\dfrac{1}{27}+\ldots\ldots+\dfrac{1}{3^n} = \dfrac{1}{2}\left[1-\dfrac{1}{3^n}\right]$
10. $\binom{5}{5}+\binom{6}{5}+\binom{7}{5}+\ldots\ldots+\binom{n+4}{5}=\binom{n+5}{6}$
11. $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots\ldots+\binom{n}{2} = \binom{n+1}{3}$ for $n \geq 2$
12. Show by mathematical induction that
    (i) $\dfrac{5^{2n}-1}{24}$ is an integer. (ii) $\dfrac{10^{n+1}-9n-10}{81}$ is an integer.
13. (i) $2^n>n \ \forall n\in N$. (ii) $n!>n^2$ for every integer $n\geq 4$
14. (i) Show that 5 is a factor of $3^{2n-1}+2^{2n-1}$ where n is any positive integer.
    (ii) Prove that $2^{2n}-1$ is a multiple of 3 for all positive integers.
15. Show that $a+b$ is a factor of a^n-b^n for all even positive integer n.

7.2 The Binomial Theorem

In algebra a sum of two terms, such as a + b, is called a binomial. The binomial theorem gives an expression for the powers of a binomial $(a + b)^n$, for each positive integer n and all real numbers a and b.

7.2.1 Statement and proof of the binomial theorem

The binomial theorem in its explicit form is stated as under.

Theorem: If a and b are any two real numbers and n is a positive integer, then

$$(a+b)^n = \binom{n}{0} a^n b^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + \binom{n}{n} a^0 b^n$$

which more compactly can be written in summation form as:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Proof: Mathematical induction provides us the best way for confirming the validity of the binomial theorem.

$$(a+b)^n = \binom{n}{o} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + \binom{n}{n} a^0 b^n \;\ldots\ldots\text{(i)}$$

Step 1. If n = 1 , then from (i), we obtain

$$(a+b)^1 = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^{1-1} b^1 = a + b \qquad \left(\because \binom{1}{0} = \binom{1}{1} = 1\right)$$

which is true. Thus the statement is true for n = 1

Step 2. Suppose that the statement is true for $n = k$, then

$$(a+b)^k = \binom{k}{0} a^k b^0 + \binom{k}{1} a^{k-1} b^1 + \binom{k}{2} a^{k-2} b^2 + \ldots + \binom{k}{r} a^{k-r} b^r + \ldots + \binom{k}{k} a^0 b^k \;\ldots\ldots\text{(ii)}$$

Step 3. We now prove that the theorem is true for $n = k + 1$. Multiplying both sides of equation (ii) by $(a + b)$, we have

$$(a+b)(a+b)^k = (a+b)\left[\binom{k}{0} a^k b^0 + \binom{k}{1} a^{k-1} b^1 + \binom{k}{2} a^{k-2} b^2 + \ldots + \binom{k}{r} a^{k-r} b^r \ldots + \binom{k}{k} a^0 b^k\right]$$

$$\Rightarrow (a+b)^{k+1} = \left[\binom{k}{0} a^{k+1} b^0 + \binom{k}{1} a^k b^1 \binom{k}{2} a^{k-1} b^2 + \ldots + \binom{k}{r} a^{k-r+1} b^r + \ldots + \binom{k}{k} a b^k\right]$$

$$+ \left[\binom{k}{0} a^k b + \binom{k}{1} a^{k-1} b^2 + \binom{k}{2} a^{k-1} b^3 + \ldots + \binom{k}{r} a^{k-r} b^{r+1} + \ldots + \binom{k}{k} a^0 b^{k+1}\right]$$

$$\Rightarrow (a+b)^{k+1} = \binom{k}{0} a^{k+1} + \left[\binom{k}{1}+\binom{k}{0}\right] a^k b + \left[\binom{k}{2}+\binom{k}{1}\right] a^{k-1}b^2 + \ldots\ldots\ldots\ldots +$$

$$\left[\binom{k}{r}+\binom{k}{r-1}\right] a^{k-r+1}\, b^r + \ldots\ldots + \binom{k}{k} a^0 b^{k+1}$$

We know that $\binom{k}{0}=\binom{k+1}{0}=1$ and $\binom{k}{k}=\binom{k+1}{k+1}, \binom{k}{r-1}+\binom{k}{r}=\binom{k+1}{r}$ for $0 \le r \le k$,

therefore,

$$(a+b)^{k+1} = \binom{k+1}{0} a^{k+1}b^0 + \binom{k+1}{1} a^k b + \binom{k+1}{2} a^{k-1}b^2 + \ldots + \binom{k+1}{r} a^{k+1-r}b^r + \ldots + \binom{k+1}{k+1} a^0 b^{k+1}$$

which is of the form (i) for $n = k + 1$

So the given statement is true for $n = k + 1$ and thus by the method of induction it is true for all positive integers n.

7.2.2 Properties of the Binomial Expansion

The expansion of $(a + b)^n$ has the following properties.

(i) The number of terms in the expansion of $(a+b)^n$ are $n+1$ i.e. the number of terms are one more than the exponent n.

Thus the expansion of $(a + b)^8$ will contain $8+1 = 9$ terms.

(ii) In the expansion of $(a + b)^n$ the first term is $a^n b^0$, the second term is $n\, a^{n-1}b^1$ and the third term is $\frac{n(n-1)}{2!} a^{n-2}b^2$ and so on . In each term the exponent of a decreases progressively by 1 and the exponent of b increases progressively by 1, but the sum of the exponents of a and b in each terms is always equal to n.

(iii) In the expansion of $(a + b)^n$ the terms $\binom{n}{r} a^{n-r}b^r$ and $\binom{n}{n-r} a^r b^{n-r}$ are equidistant from the beginning and the end. For $\binom{n}{r} a^{n-r}b^r$ is preceeded by r terms and followed by $n - r$ terms while $\binom{n}{n-r} a^r b^{n-r}$ is preceded by $n - r$ terms and followed by r terms. Also since $\binom{n}{n-r} = \frac{n!}{(n-r)!r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$

So the coefficients of terms equidistant from the beginning and end are equal.

(iv) In the expansion of $(a + b)^n$, if n is even, the number of terms are odd and there will be only one middle term. If n is odd, the number of terms are even and there will be two middle terms.

(v) For n even in $(a+b)^n$, the $\left(\frac{n+2}{2}\right)$th term is the only one middle term and for n odd the $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms are the two middle terms.

(vi) In $(a+b)^n$ if b is replaced by $-b$ then $(a-b)^n$ has expansion of the form

$$(a-b)^n = \binom{n}{0} a^n b^0 - \binom{n}{1} a^{n-1}b + \binom{n}{0} a^{n-2}b^2 + \ldots\ldots\ldots + (-1)^n \binom{n}{n} a^0 b^n$$

or $(a-b)^n = a^n - \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots\ldots\ldots + (-1)^n b^n$.

We note that in the expansion of $(a-b)^n$ the terms are alternately positive and negative.

(vii) In the expansion of $(a+b)^n$ the (r+1)th term which is $\binom{n}{r} a^{n-r}b^r$ is usually called the **general term** and is denoted by T_{r+1}.

Thus $T_{r+1} = \binom{n}{r} a^{n-r}b^r = \frac{n!}{r!(n-r)!} a^{n-r}.b^r$

We note that for using binomial formula for given value of n, in the expansion of $(a+b)^n$, the most important task is to find the binomial coefficients

$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots\ldots$etc.

7.2.3 Pascal's Triangle

Consider the following expanded powers of $(a+b)^n$, where $a+b$ is any binomial and n is a whole number. Look for patterns.

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Each expansion is a polynomial. There are some patterns to be noted.

(i) There is one more term than the power of the exponent, n. That is, there are $n+1$ terms in the expansion of $(a+b)^n$.

(ii) In each term, the sum of the exponents is n, the power to which the binomial is raised.

(iii) The exponents of a start with n, the power of the binomial, and decrease to 0. The last term has no factor of a. The first term has no factor of b, so powers of b start with 0 and increase to n, (iv)The coefficients start at 1 and increase through certain values about "half-way" and then decrease through these same values back to 1.

The above binomial expansions can be written in the following triangular form

$$\begin{array}{c} 1 \\ a+b \\ a^2+2ab+b^2 \\ a^3+3a^2b+3ab^2+b^3 \\ a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\ a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \end{array}$$

Did You Know

Pascal's triangle is most convenient to obtain the coefficients of the binomial expansion $(a+b)^n$ when n is a small number.

For each of the above expansions, we write down the binomial coefficients in the following fashion

n	Values of binomial coefficients
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

The above configuration of numbers is called **Pascal's Triangle.**

Example 8: Find the expansion of $(x+y)^6$.

Solution: By the formula,

$$(x+y)^6 = x^6 + {}^6C_1x^5y + {}^6C_2x^4y^2 + {}^6C_3x^3y^3 + {}^6C_4x^2y^5 + {}^6C_5xy^5 + {}^6C_6y^6$$

$$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6.$$

On calculating the value of ${}^6C_1$, ${}^6C_2$, ${}^6C_3$,........

Example 9: Find the 6th term in the expansion of $(3x+2y)^{12}$.

Solution: Let T_{r+1}th term be the sixth term of the expansion $(3x+2y)^{12}$. We remember that the T_{r+1}th term for the expansion of $(a+b)^n$ is $T_{r+1}=\binom{n}{r}a^{n-r}b^r$

So, for the given expansion $(3x+2y)^{12}$

$T_{r+1}=\binom{12}{r}(3x)^{12-r}(2y)^r$. Here we have $n=12$, $a=3x$ and $b=2y$

Since we are interested in finding the 6th term i.e. T_6, so choosing r = 5 and putting in the last result , we have,

$$T_{5+1} = T_6 = \binom{12}{5}(3x)^{12-5} \cdot (2y)^5 \qquad \Rightarrow T_6 = \frac{12!}{5!\,7!}3^7 \cdot x^7 \cdot 2^5 \cdot y^5$$

$$\Rightarrow T_6 = \frac{12.11.10.9.8.7!}{1.2.3.4.5.7!} \times 2187 \times 32x^7 y^5 \qquad \Rightarrow T_6 = 11 \cdot 9 \cdot 8 \cdot 2187 \cdot 32x^7y^5$$

$$\Rightarrow T_6 = 55427328\, x^7y^5$$

Example 10: Find the coefficient of x^5 in the expansion of $(2x^2 - \frac{3}{x})^{10}$

Solution: Let T_{r+1} of $(2x^2 - \frac{3}{x})^{10}$ be the particular terms containing x^5.

Now for the given expansion $(2x^2 - \frac{3}{x})^{10}$

$$T_{r+1} = \binom{10}{r}(2x^2)^{10-r}\left(-\frac{3}{x}\right)^r = \binom{10}{r}2^{10-r}(x^2)^{10-r}(-1)^r \cdot \frac{3^r}{x^r}$$

$$= (-1)^r\binom{10}{r}2^{10-r} \cdot 3^r x^{20-2r} \cdot x^{-r} = (-1)^r\binom{10}{r}2^{10-r} \cdot 3^r \cdot x^{20-2r-r}$$

$$T_{r+1} = (-1)^r\binom{10}{r}2^{10-r} \cdot 3^r \cdot x^{20-3r} \qquad (1)$$

But this term contains x^5 and this is only possible if $x^{20-3r} = x^5$ and thus $20-3r = 5$
$\Rightarrow 3r = 20 - 5$ or $3r = 15 \Rightarrow r = 5$ Putting this value of $r = 5$ in (1) we get.

$$T_{5+1} = T_6 = (-1)^5\binom{10}{5}2^{10-r} \cdot 3^5 x^{20-15} \qquad \Rightarrow T_6 = (-1)^5\binom{10}{5}2^5 \cdot 3^5 x^5$$

So the required coefficient is $(-1)^5\binom{10}{5}2^5 \cdot 3^5 = -\frac{10!}{5!\,5!}32 \cdot 243$

$\therefore$ Required coefficient $= -\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 5!} \cdot 32 \cdot 243$, that is the required coefficient of $x^5 = -1959552$

Example 11: Find the term independent of x in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

Solution: Let T_{r+1}th term of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ be the particular term which is independent of x. The T_{r+1}th term for the above expansion is

$$T_{r+1} = \binom{9}{r}\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r = \binom{9}{r}\left(\frac{3}{2}\right)^{9-r} x^{2(9-r)}(-1)^r.\frac{1}{3^r}.\frac{1}{x^r}$$

$$= (-1)^r\binom{9}{r}\left(\frac{3}{2}\right)^{9-r}\frac{1}{3^r}.x^{18-2r}.x^{-r} = (-1)^r\binom{9}{r}\left(\frac{3}{2}\right)^{9-r}\frac{1}{3^r}x^{18-2r-r}$$

$$= (-1)^r\binom{9}{r}\left(\frac{3}{2}\right)^{9-r}\frac{1}{3^r}x^{18-3r} \qquad (1)$$

But T_{r+1} th term is free of x and this is possible if $x^{18-3r} = x^0$ giving $18 - 3r = 0$ $\Rightarrow 3r = 18$ and so $r = 6$

Thus $T_{r+1} = T_{6+1} = T_7$ i.e. 7th term of the given expansion is independent of x.

$$T_7 = (-1)^6\binom{9}{6}\left(\frac{3}{2}\right)^{9-6}\frac{1}{3^6}x^{18-18} = \frac{9!}{6!\,3!}\frac{3^3}{2^3}\cdot\frac{1}{3^6}\cdot 1$$

$$= \frac{9\cdot8\cdot7\cdot6!}{6!\,2\cdot3}\cdot\frac{1}{2^3}\cdot\frac{1}{3^3} = \frac{9\cdot8\cdot7\cdot}{2\cdot3}\cdot\frac{1}{8}\cdot\frac{1}{27} = \frac{7}{18}$$

Thus the 7th term of the expansion $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is independent of x and its value is $\frac{7}{18}$.

Example 12: Find the middle term in the expansion of $\left(\frac{a}{x}+\frac{x}{a}\right)^{10}$.

Solution: Since in $\left(\frac{a}{x}+\frac{x}{a}\right)^{10}$, n = 10 which is even, so that total number of terms in the above expansion = 10+1 = 11. Thus it has only one middle term which is $\left(\frac{n+2}{2}\right)$th term = $\left(\frac{10+2}{2}\right)$th term = 6th term.i.e. 6th term is the middle term

Now T_{r+1} for $\left(\frac{a}{x}+\frac{x}{a}\right)^{10}$ is given by

$$T_{r+1} = \binom{10}{r}\left(\frac{a}{x}\right)^{10-r}\left(\frac{x}{a}\right)^r. \text{ Putting } r = 5$$

We get $T_6 = \binom{10}{5}\left(\frac{a}{x}\right)^5\left(\frac{x}{a}\right)^5 = \frac{10!}{5!\,5!}\cdot\left(\frac{a^5}{x^5}\right)\left(\frac{x^5}{a^5}\right) = \frac{10\cdot9\cdot8\cdot7\cdot6\cdot5!}{5!\cdot1\cdot2\cdot3\cdot4\cdot5} = 252.$

So the 6th term of $\left(\frac{a}{x}+\frac{x}{a}\right)^{10}$ is the middle term and it is 252.

EXERCISE 7.2

1. Expand by using Binomial theorem.

   (i) $\left(x^2-\frac{1}{y}\right)^4$ (ii) $(1+xy)^7$ (iii) $\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right)^5$

2. Find the indicated term in the expansions.

   (i) 4^{th} term in $(2+a)^7$ (ii) 8^{th} term in $\left(\frac{x}{2}-\frac{3}{y}\right)^{10}$ (iii) 3^{rd} term in $\left(x^2+\frac{1}{\sqrt{x}}\right)^4$

3. Find the term independent of x in the following expansions.

   (i) $\left(\frac{4a^2}{3}-\frac{3}{2a}\right)^9$ (ii) $\left(x-\frac{3}{x^4}\right)^{10}$ (iii) $\left(x-\frac{1}{x^2}\right)^{21}$

4. Find the coefficient of

   (i) x^{23} in $\left(x^2-x\right)^{20}$ (ii) $\frac{1}{x^4}$ in $\left(2-\frac{1}{x}\right)^8$ (iii) a^6b^3 in $\left(2a-\frac{b}{3}\right)^9$

5. Find the middle term in the expansion of:

   (i) $\left(\frac{a}{x}+bx\right)^8$ (ii) $\left(3x-\frac{x^2}{2}\right)^9$ (iii) $\left(3x^2-\frac{y}{3}\right)^{10}$

6. Find the constant term in the expansion of $\left(2\sqrt{x}-\frac{3}{x\sqrt{x}}\right)^{23}$.

7. Find

   (i) $\left(2+\sqrt{3}\right)^5+\left(2-\sqrt{3}\right)^5$ (ii) $\left(1+\sqrt{2}\right)^4-\left(1-\sqrt{2}\right)^4$ (iii) $(a+b)^5+(a-b)^5$

8. Find the numerically greatest term in $(3-2x)^{10}$, when $x=\frac{3}{4}$.

9. Find the numerically greatest term in the expansion of $(x-y)^{20}$ when $x=12$ and $y=4$.

10. Prove that sum of Binomial coefficients of order $n = 2^n$. Also prove the sum of odd binomial coefficients = sum of even Binomial coefficients $= 2^{n-1}$.

11. Consider $(1+x)^n$ and take $\binom{n}{r}=C_r$.

    Show that $C_1+2C_2x+3C_3x^2+\ldots\ldots+nC_nx^{n-1}=n(1+x)^{n-1}$

7.3 Binomial Series

7.3.1 Expansion of $(1+x)^n$ where n is a positive integer

By Binomial theorem, for any two real numbers a and b and for a positive integer n

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots\ldots + b^n \qquad \text{(i)}$$

and this expansion contains (n + 1) terms. Now in particular if $a = 1$ and $b = x$ then the above expansion becomes

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots + x^n \qquad \text{(ii)}$$

Thus we observe that when n is a positive integer then the binomial expansion $(a+b)^n$ or $(1+x)^n$ terminates after (n + 1)th term.

7.3.2 Expansion of $(1+x)^n$ where n, the exponent, is a negative integer or a fraction

If n is a negative integer or a fraction, then the expansion (ii) never ends and thus in such a case the expansion becomes

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots \qquad \text{(iii)}$$

When n is a negative integer or a fraction then the series as given in (iii) is convergent if $-1<x<1$ or $|x|<1$ and it is divergent if $|x|>1$.

Since at this level we will be interested only in those series which are convergent so we will say that if n is a negative integer or a fraction then the series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots \text{ is valid only if } |x|<1.$$

The series of the type $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$ is called the **binomial series**.

The general term of the binomial series is

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}x^r$$

Example 13: Find the first four terms in the expansion of $(1+x)^{-\frac{1}{2}}$

Solution: $(1+x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)x + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}x^2 + \dfrac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}x^3 + \dots\dots$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2}x^2 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}x^3 + \dots\dots\dots$$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{1}{2}.\frac{3}{2}.\frac{1}{2}x^2 - \frac{1}{2}.\frac{3}{2}.\frac{5}{2}.\frac{1}{6}x^3 + \dots\dots$$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots\dots\dots$$

Example 14: Find the first four terms in the expansion of $\left(9+\frac{4}{x}\right)^{\frac{1}{2}}$ for $|x| > \frac{4}{9}$.

Solution:

$$\left(9+\frac{4}{x}\right)^{\frac{1}{2}} = \left(9\left(1+\frac{4}{9x}\right)\right)^{\frac{1}{2}} = 9^{\frac{1}{2}}\left(1+\frac{4}{9x}\right)^{\frac{1}{2}} = 3\left(1+\frac{4}{9x}\right)^{\frac{1}{2}}$$

$$= 3\left[1 + \frac{1}{2}.\frac{4}{9x} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{4}{9x}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{4}{9x}\right)^3 + \dots\dots\right]$$

$$= 3\left[1 + \frac{2}{9x} - \frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{16}{81x^2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\left(\frac{64}{729x^3}\right) + \dots\dots\right]$$

$$= 3\left[1 + \frac{2}{9x} - \frac{2}{81x^2} + \frac{3}{8\times 6}.\frac{64}{729x^3} + \dots\dots\right]$$

$$= 3\left[1 + \frac{2}{9x} - \frac{2}{81x^2} + \frac{4}{729x^3} + \dots\dots\right]$$

Example 15: Compute $\sqrt[3]{\frac{5}{4}}$ to an accuracy of at least four decimal places using binomial expansion

Solution:

Given $\sqrt[3]{\frac{5}{4}} = \left(\frac{5}{4}\right)^{\frac{1}{3}} = \left(\frac{4+1}{4}\right)^{\frac{1}{3}}$ or

$$\left(\frac{5}{4}\right)^{\frac{1}{3}} = \left(1+\frac{1}{4}\right)^{\frac{1}{3}} = 1+\frac{1}{3}.\frac{1}{4}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{4}\right)^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\left(\frac{1}{4}\right)^3+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{4!}\left(\frac{1}{4}\right)^4+\ldots\ldots$$

$$\text{or } \left(\frac{5}{4}\right)^{\frac{1}{3}} = 1+\frac{1}{12}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}\cdot\frac{1}{16}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}\cdot\frac{1}{64}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{24}\cdot\frac{1}{256}+\ldots\ldots$$

$$= 1+\frac{1}{12}-\frac{1}{9}.\frac{1}{16}+\frac{1}{3}.\frac{2}{3}.\frac{5}{3}.\frac{1}{6}.\frac{1}{64}-\frac{1}{3}.\frac{2}{3}.\frac{5}{3}.\frac{8}{3}.\frac{1}{24}.\frac{1}{256}+\ldots\ldots\ldots\ldots$$

$$= 1+\frac{1}{12}-\frac{1}{144}+\frac{5}{81\times64}-\frac{5}{2.3^5.4^3}+\ldots\ldots\ldots\ldots$$

$$= 1 + 0.08333 - 0.00694 + 0.00096 - 0.000016 + \ldots\ldots\ldots\ldots$$

Taking only these five terms and neglecting the other we can write

$$\sqrt[3]{\frac{5}{4}} \approx 1.00000+0.08333-0.00694+0.00096-0.000016.$$

Where $\approx$ stands for 'approximately equal to'. We have used here the symbol $\approx$ because we have omitted all the terms after the first five terms. So we cannot expect even think for exactness.

$$\sqrt[3]{\frac{5}{4}} \approx 1.07719 \approx 1.0772$$

Example 16: Evaluate $\sqrt{35}$ by Binomial theorem

Solution:

$$\sqrt{35} = (36-1)^{\frac{1}{2}} = \left[36\left(1-\frac{1}{36}\right)\right]^{\frac{1}{2}} = (36)^{\frac{1}{2}}\left[1-\frac{1}{36}\right]^{\frac{1}{2}}$$

$$= 6\left[1-\frac{1}{2}.\frac{1}{36}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{36}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{1}{36}\right)^3+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}\left(-\frac{1}{36}\right)^4+\ldots\ldots\right]$$

$$\sqrt{35}=6\left[1-\frac{1}{72}-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{1296}-\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{6}\cdot\frac{1}{46656}-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{24}\cdot\frac{1}{1679616}\cdots\cdots\right]$$

$$=6\left[1-\frac{1}{72}-\frac{1}{10368}-\frac{1}{746496}-\frac{5}{214990848}-\cdots\cdots\cdots\right]$$

$$\approx 6[1-0.013888-0.00009645-0.000001339]$$

$$\sqrt{35}\approx 6[0.9860142]$$

$$\sqrt{35}\approx 5.9160852$$

$$\sqrt{35}\approx 5.9161$$

Example 17: If x be so small that its square and higher powers may be neglected then evaluate

(i) $\dfrac{\sqrt{4+x}}{4-\frac{x}{3}}$ (ii) $\dfrac{(1+x)^{\frac{1}{2}}(16-5x)^{\frac{1}{2}}}{(9+2x)^{\frac{1}{2}}}$

Solution:

(i) $$\frac{\sqrt{4+x}}{4-\frac{x}{3}}=\frac{\sqrt{4\left(1+\frac{x}{4}\right)}}{4\left(1-\frac{x}{12}\right)}=\frac{2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}}{4\left(1-\frac{x}{12}\right)}=\frac{1}{2}\left(1+\frac{x}{4}\right)^{\frac{1}{2}}\left(1-\frac{x}{12}\right)^{-1}$$

$$=\frac{1}{2}\left(1+\frac{1}{2}\cdot\frac{x}{4}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x}{4}\right)^2+\cdots\cdots\cdots\right)\times\left(1+\frac{x}{12}+\frac{x^2}{144}+\cdots\cdots\cdots\right)$$

$$=\frac{1}{2}\left(1+\frac{x}{8}-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{x^2}{16}+\cdots\cdots\cdots\right)\left(1+\frac{x}{12}+\frac{x^2}{144}+\cdots\cdots\cdots\right)$$

$$=\frac{1}{2}\left[1+\frac{x}{12}+\frac{x}{8}+\text{ Ignoring terms containing } x^2,x^3,\cdots\cdots\cdots\right]$$

$$=\frac{1}{2}\left[1+\frac{2x+3x}{24}\right]=\frac{1}{2}\left[1+\frac{5x}{24}\right]$$

$$\therefore \frac{\sqrt{4+x}}{4-\frac{x}{3}}=\frac{1}{2}\left[1+\frac{5x}{24}\right]$$

where x is so small such that x^2 and higher powers are neglected.

(ii) Now taking $\dfrac{(1+x)^{\frac{1}{2}}(16-5x)^{\frac{1}{2}}}{(9+2x)^{\frac{1}{2}}} = \dfrac{(1+x)^{\frac{1}{2}}(16)^{\frac{1}{2}}\left(1-\frac{5}{16}x\right)^{\frac{1}{2}}}{9^{\frac{1}{2}}\left(1+\frac{2x}{9}\right)^{\frac{1}{2}}}$

$$= \frac{4(1+x)^{\frac{1}{2}}\left(1-\frac{5}{16}x\right)^{\frac{1}{2}}}{3\left(1+\frac{2x}{9}\right)^{\frac{1}{2}}} = \frac{4}{3}(1+x)^{\frac{1}{2}}\left(1-\frac{5}{16}x\right)^{\frac{1}{2}}\left(1+\frac{2x}{9}\right)^{-\frac{1}{2}}$$

$$= \frac{4}{3}\left[1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+......\right]\left[1-\frac{1}{2}.\frac{5}{16}x+......\right]\times\left[1-\frac{1}{2}.\frac{2x}{9}+......\right]$$

$$= \frac{4}{3}\left[1+\frac{x}{2}+..........\right]\left[1-\frac{5}{32}x+....................\right]\left[1-\frac{x}{9}+..........\right]$$

$$= \frac{4}{3}\left[1+\frac{x}{2}+..........\right]\left[1-\frac{x}{9}-\frac{5}{32}x+\frac{5}{32}.\frac{1}{9}x^2..............\right]$$

$$= \frac{4}{3}\left[1+\frac{x}{2}\right]\left(1-\frac{32x+45x}{9\times 32}\right)$$ Ignoring terms containing $x^2, x^3, x^4,\ldots$

$$= \frac{4}{3}\left[1+\frac{x}{2}\right]\left(1-\frac{77x}{9\times 32}\right) = \frac{4}{3}\left(1-\frac{77x}{9\times 32}+\frac{x}{2}\right)$$ Again ignoring term containing x^2

or $$\frac{(1+x)^{\frac{1}{2}}(16-5x)^{\frac{1}{2}}}{(9+2x)^{\frac{1}{2}}} = \frac{4}{3}\left(1+\frac{x}{2}-\frac{77x}{9\times 32}\right) = \frac{4}{3}\left(1+\frac{9\times 16x-77x}{9\times 32}\right) = \frac{4}{3}\left(1+\frac{144x-77x}{288}\right)$$

$$\Rightarrow \quad \frac{(1+x)^{\frac{1}{2}}(16-5x)^{\frac{1}{2}}}{(9+2x)^{\frac{1}{2}}} = \frac{4}{3}\left(1+\frac{67x}{288}\right)$$

7.4 Application of the Binomial Theorem

Approximations: We have seen in the particular cases of the expansion of $(1+x)^n$ that the power of x go on increasing in each expansion. Since $|x|<1$, so $|x|^r < |x|$ *for* 2,3,4...

This fact shows that terms in each expansion go on decreasing numerically if $|x|<1$.

Thus some initial terms of the binomial series are enough for determining the approximate values of binomial expansions having indices as negative integers or fractions.

Summation of infinite series: The binomial series are conveniently used for summation of infinite series. The series (*whose sum is required*) is compared with

$$1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots$$

to find out the values of n and x. Then the sum is calculated by putting the values of n and x in $(1+x)^n$.

Example 18: Find the sum of the series $1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2\cdot5}{3\cdot6}\cdot\frac{1}{2^2}+\frac{2\cdot5\cdot8}{3\cdot6\cdot9}\cdot\frac{1}{2^3}\dots\dots\dots$

Solution: Suppose that the given series is identical with the expansion $(1+x)^n$.

We have $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}\cdot x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots\dots$ (i)

$$S = 1 + \frac{2}{3}\cdot\frac{1}{2}+\frac{2\cdot5}{3\cdot6}\cdot\frac{1}{2^2}+\frac{2\cdot5\cdot8}{3\cdot6\cdot9}\cdot\frac{1}{2^3}+\dots\dots \quad \text{(ii)}$$

Comparing (i) and (ii),we get

$$nx=\frac{2}{3}\cdot\frac{1}{2} \quad and \quad \frac{n(n-1)}{2!}x^2=\frac{2\cdot5}{3\cdot6}\cdot\frac{1}{2^2}$$

Squaring $n^2x^2=\frac{1}{9}$ and $\frac{n(n-1)}{2!}x^2=\frac{5}{36}$ so that

$$\frac{\frac{n(n-1)}{2!}x^2}{n^2x^2}=\frac{\frac{5}{36}}{\frac{1}{9}} \Rightarrow \frac{n(n-1)}{2}\cdot\frac{1}{n^2}=\frac{5}{36}\times\frac{9}{1} \Rightarrow \frac{n-1}{2n}=\frac{5}{4}\Rightarrow\frac{n-1}{n}=\frac{5}{2}$$

$$\Rightarrow 5n=2n-2 \Rightarrow 5n-2n=-2 \Rightarrow 3n=-2\Rightarrow n=-\frac{2}{3}$$

Putting this value of n in $nx=\frac{2}{3}\cdot\frac{1}{2}$

We get $-\frac{2}{3}x=\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{3} \quad \Rightarrow -2x=1\Rightarrow x=-\frac{1}{2}$

So, $S=(1+x)^n=(1-\frac{1}{2})^{\frac{-2}{3}}=\left(\frac{1}{2}\right)^{\frac{-2}{3}}=\frac{1}{\left(\frac{1}{2}\right)^{\frac{2}{3}}}=\frac{1}{\frac{1}{2^{\frac{2}{3}}}}=2^{\frac{2}{3}}=4^{\frac{1}{3}}$

i e. $S = 4^{\frac{1}{3}}$ and so from (ii)

$$1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2\cdot5}{3\cdot6}\cdot\frac{1}{2^2}+\frac{2\cdot5\cdot8}{3\cdot6\cdot9}\cdot\frac{1}{2^3}+\dots\dots\dots=4^{\frac{1}{3}}$$

Example 19: If $y = \frac{3}{4} + \frac{3\cdot5}{4\cdot5} + \frac{3\cdot5\cdot7}{4\cdot8\cdot12} + \ldots\ldots\ldots\ldots\ldots\ldots$ Show that $y^2 + 2y - 7 = 0$

Solution: Given that $y = \frac{3}{4} + \frac{3\cdot5}{4\cdot8} + \frac{3\cdot5\cdot7}{4\cdot8\cdot12} + \ldots\ldots\ldots\ldots\ldots$

$$\Rightarrow y + 1 = 1 + \frac{3}{4} + \frac{3\cdot5}{4\cdot8} + \frac{3\cdot5\cdot7}{4\cdot8\cdot12} + \ldots\ldots\ldots\ldots(1)$$

Let the series on the R.H.S. of (1) be identical with the expansion $(1 + x)^n$.
We have,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots\ldots(2)$$

Comparing right hand sides of (1) and (2), we have,

$nx = \frac{3}{4}$ ....(3) *and* $\frac{n(n-1)}{2}x^2 = \frac{3\cdot5}{4\cdot8}$ ....(4) Squaring equation (3)

$n^2x^2 = \frac{9}{16}$ ....(5) Dividing equation (4) by equation (5)

$$\frac{\frac{n(n-1)x^2}{2}}{n^2x^2} = \frac{\frac{3\cdot5}{4\cdot8}}{\frac{9}{16}} \quad \text{or} \quad \frac{n(n-1)x^2}{2} \times \frac{1}{n^2x^2} = \frac{3\cdot5}{4\cdot8} \times \frac{16}{9}$$

$$\Rightarrow \frac{n(n-1)}{2n^2} = \frac{5}{6} \Rightarrow \frac{n-1}{n} = \frac{5}{3}$$

or $3(n-1) = 5n \Rightarrow 3n - 3 = 5n \Rightarrow -3 = 5n - 3n \Rightarrow 2n = -3$

$\Rightarrow n = -\frac{3}{2}$ Putting $n = -\frac{3}{2}$ in $nx = \frac{3}{4}$, we get $\left(-\frac{3}{2}\right)x = \frac{3}{4} \Rightarrow -\frac{x}{2} = \frac{1}{4}$

$x = \frac{-2}{4}$ or $x = -\frac{1}{2}$.

So $y + 1 = (1 + x)^n = 1 + \frac{3}{4} + \frac{3\cdot5}{4\cdot8} + \frac{3\cdot5\cdot7}{4\cdot8\cdot12} + \ldots\ldots\ldots\ldots$becomes

$y + 1 = (1 - \frac{1}{2})^{\frac{-3}{2}}$ or $y + 1 = (\frac{1}{2})^{\frac{-3}{2}} \Rightarrow (y + 1)^2 = (\frac{1}{2})^{-3}$

or $(y+1)^2 = \frac{1}{\left(\frac{1}{2}\right)^3} = \frac{1}{\frac{1}{8}}$

i.e. $(y+1)^2 = 8 \Rightarrow y^2 + 2y + 1 - 8 = 0 \Rightarrow y^2 + 2y - 7 = 0$

EXERCISE 7.3

1. Find the first four terms in the expansions of

(i) $(1-x)^{\frac{-1}{2}}$ (ii) $(1-x)^{\frac{3}{2}}$ (iii) $(8+12x)^{\frac{2}{3}}$

2. (i) Find $\sqrt{26}$ correct to 3 decimal places.

(ii) Evaluate $\dfrac{1}{\sqrt{(.998)}}$ to four significant figures.

(iii) Find the cube root of 126 correct to five decimal places.

3. Expand: $\sqrt{\dfrac{1-x}{1+x}}$ up to x^3.

4. If x is such that x^2 and higher powers may be neglected, then show that $\sqrt{\dfrac{1-3x}{1+4x}} = 1-\dfrac{7x}{2}$

5. If x is so small that its square and higher powers can be neglected, then show that

$$\frac{(8+3x)^{\frac{2}{3}}}{(2+3x)\sqrt{4-5x}} = 1-\frac{5}{8}x$$

6. If x is large and if $\dfrac{1}{x^3}$ may be neglected, then find the approximate value of: $\dfrac{x\sqrt{x^2-2x}}{(x+1)^2}$

7. If x^4 and higher powers are neglected, such that

$(1+x)^{\frac{1}{4}} + (1-x)^{\frac{1}{4}} = a - bx^2$. Find a and b.

8. If x is of such a size that its values are considered up to x^3.

Show that: $\dfrac{(1+\frac{1}{2}x)^3-(1+3x)^{\frac{1}{2}}}{1-\frac{5}{6}x} = \dfrac{15x^2}{8}$

9. Find the co-efficient of x^n in $\left(\dfrac{1+x}{1-x}\right)^2$

10. Find the sum of the following:

(i) $1-\dfrac{1}{2^2}+\dfrac{1\cdot3}{2!}\cdot\dfrac{1}{2^4}+\ldots\ldots$ (ii) $1+\dfrac{5}{8}+\dfrac{5\cdot8}{8\cdot12}+\dfrac{5\cdot8\cdot11}{8\cdot12\cdot16}+\ldots\ldots$

11. If $y = \frac{1}{2^2} + \frac{1\cdot3}{2!}\cdot\frac{1}{2^4} + \frac{1\cdot3\cdot5}{3!}\cdot\frac{1}{2^6} + \ldots\ldots$ then $y^2 + 2y - 1 = 0$

12. If $2y = \frac{1}{2^2} + \frac{1\cdot3}{2!}\cdot\frac{1}{2^4} + \frac{1\cdot3\cdot5}{3!}\cdot\frac{1}{2^6} + \ldots\ldots$ then $4y^2 + 4y - 1 = 0$

13. If x is so small that x^3 and higher powers of x can be ignored. Show that the nth root of $1 + x$ is equal to $\frac{2n+(n+1)x}{2n+(n-1)x}$

14. If x is nearly equal to unity then show that $px^p - qx^q = (p - q)\,x^{p+q}$

REVIEW EXERCISE 7

1. (i) What is the middle term in the expansion of $(2x+5y)^4$?

   (a) $600\,x^2y^2$ (b) $120\,xy^2$ (c) $5000\,xy^3$ (d) $6\,x^2y^2$

   (ii) What is the coefficient of the term containing $x^{12}y^6$ in the expansion of $(x^3 - 2y^2)^7$?

   (a) 84 (b) –280 (c) 560 (d) 448

   (iii) The expansion of $\left(x+\sqrt{x^2-1}\right)^5 + \left(x-\sqrt{x^2-1}\right)^5$ is a polynomial of degree

   (a) 5 (b) 6 (c) 7 (d) 8

   (iv) Number of terms in expansion of $\left(\sqrt{x}+\sqrt{y}\right)^{10} + \left(\sqrt{x}-\sqrt{y}\right)^{10}$ is

   (a) 6 (b) 11 (c) 20 (d) 5

   (v) $\left(\sqrt{2}+1\right)^5 + \left(\sqrt{2}-1\right)^5 =$ ---------------

   (a) 58 (b) $58\sqrt{2}$ (c) -58 (d) $-58\sqrt{2}$

   (vi) $\binom{n-1}{1} + \binom{n-1}{2} + \ldots\ldots\ldots\ldots + \binom{n-1}{n-1} = \ldots\ldots,\ n > 1$

   (a) $2^n - 1$ (b) 2^{n-2} (c) $2^{n-1} - 1$ (d) 2^{n-1}

   (vii) Sum of coefficients of last 15 terms in expansion of $(1+x)^{29}$ is

   (a) 2^{15} (b) 2^{30} (c) 2^{29} (d) 2^{28}

   (viii) ${}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + \ldots\ldots\ldots\ldots + {}^{10}C_9 =$ -------------

   (a) 512 (b) 1024 (c) 2048 (d) 1023

2. Find the middle term in the expansion of $(2x^3 + 3y)^8$

3. What is the coefficient of the fourth term in the expansion of $(2x - 4y)^7$?

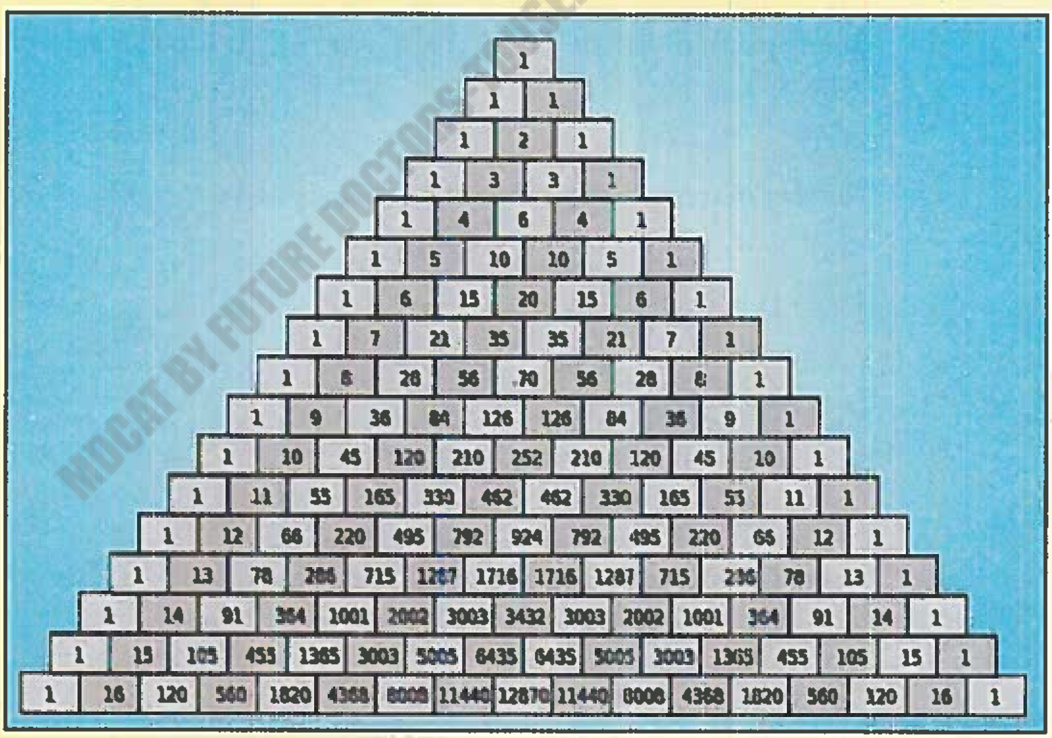
4. 2^7xy^3 is a term in the expansion of $(ax + 2y)^4$. Find a.

5. What is the constant term in the expansion of $\left(\frac{2}{x^2}+\frac{x^2}{2}\right)^{10}$

6. Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

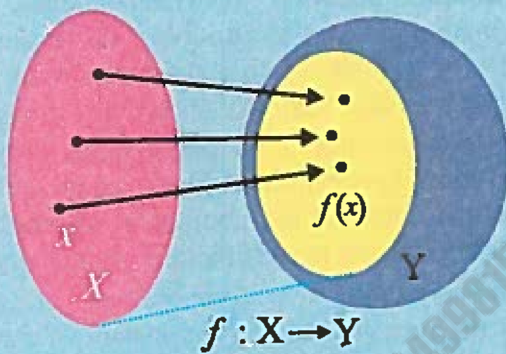
7. For every positive integer n, prove that $7^n - 3^n$ is divisible by 4.

8. Prove that $(1+x)^n \geq (1+nx)$, for all natural number n where $x>-1$



UNIT 8

FUNCTIONS AND GRAPHS



STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Recall
  - function as a rule of correspondence,
  - domain, co-domain and range of a function,
  - one to one and onto functions.
- Know linear, quadratic and square root functions.
- Define inverse functions and demonstrate their domain and range with examples.
- Sketch graphs of
  - linear functions (e.g. $y = ax + b$),
  - non-linear functions (e.g. $y = x^2$).
- Sketch the graph of the function $y = x^n$ where n is
  - a + ve integer,
  - a −ve integer (x),
  - a rational number for $x > 0$.
- Sketch graph of quadratic function of the form $y = ax^2 + bx + c$, $a(\neq 0)$, b, c are integers.
- Sketch graph using factors.
- Predict functions from their graphs (use the factor form to predict the equation of a function of the type $f(x) = ax^2 + bx + c$, if two points where the graph crosses x-axis and third point on the curve, are given).
- Find the intersecting point graphically when intersection occurs between
  - a linear function and coordinate axes,
  - two linear functions,
  - a linear and a quadratic function.
- Solve, graphically, appropriate problems from daily life.

8.1. Introduction

In many practical situations the value of one quantity depends on the value of another quantity. Such dependence of one quantity on another is described mathematically as **function**. For example, one of the indicators on the dashboard of a car shows that the amount of petrol in gallons in the tank is decreasing and another indicator shows that the distance travelled in **kilometers** is increasing. In this example, we observe that there are two variable quantities and there is a relation between them. The variable quantities are the number of gallons of petrol in the tank and the number of **kilometers** travelled. Thus, the distance travelled in **kilometers** is the function of numbers of gallon of petrol in the tank.

As another example, the temperature of air throughout the day depends on the instant of time, so we can say that temperature of air is a function of instant of time. In general, if a variable denoted by y (say) is associated in a definite way with a variable x, then y is said to be a function of x.

Remember

A constant is a symbol that always represents the same number, on the other hand, A variable is a symbol that may represents different values in the same problem.

To be more specific,"If the values of y depend on x in such a way that each value of x determines exactly one and only one value of y, then y is a function of x".

Symbolically, we write $y = f(x)$. (1)

Which reads as"y is a function of x or simply y is equal to f of x". In equation (1) the variable x is called the **independent variable** (or **argument**) whereas y is called the **dependent variable.**

8.1.1 Function as a rule or correspondence

In this section, we give formal definition of a function.

A function from a set X to a set Y is a rule or correspondence that assigns to each element x in X a unique element y in Y. Symbolically, we write it as f: X→Y and read as "f is a function from X to Y".

The elements of X are called **pre-images** and the corresponding elements of Y are called the **images.** If $y \in Y$ is an image of $x \in X$ under the functions f, we write it as $y = f(x)$. Equivalently, we say that y is the **value** of the function f at x, see (Figure 8.1).

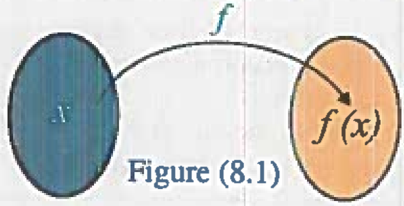


Figure (8.1)

Illustration: The following is a function, which relates the time of day to the temperature.

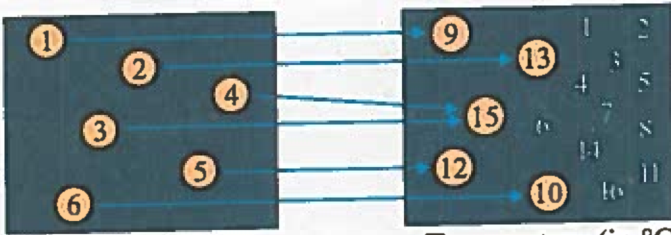


Figure (8.2)

Example 1: Let $X=\{a,b,c\}$ and $Y=\{1,2,3\}$. State whether or not the rules indicated by the following figures are functions from X to Y.

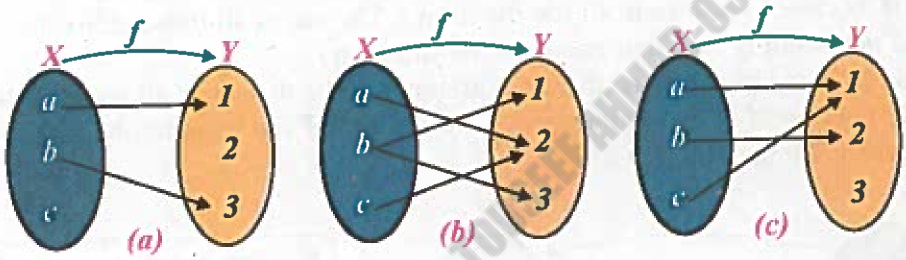


Figure (8.3)

Solution:

(1) The figure *(a)* does not define a function, because the element c of the set X has not been assigned any element of Y.

(2) The figure *(b)* does not define a function, because the element b of X has been assigned two elements of Y.

(3) The figure *(c)* does define a function, because every element of X has been assigned a unique element of Y. It may be noted that definition of function does not require that each element of Y should be an image of some element of X.

Example 2: Evaluating a function

Let $g(x)=-x^2+4x+1$.

Find each function value. a. $g(2)$ b. $g(t)$ c. $g(x+2)$

Solution:

a. Replacing x with 2 in $g(x)=-x^2+4x+1$ yields the following.

$$g(2)=-(2)^2+4(2)+1=-4+8+1=5$$

b. Replacing x with t yields the following.

$$g(t)=-(t)^2+4(t)+1=-t^2+4t+1$$

c. Replacing x with $x+2$ yields the following.

$$\begin{aligned} g(x+2) &= -(x+2)^2+4(x+2)+1 \\ &= -\left(x^2+4x+4\right)+4x+8+1 \\ &= -x^2-4x-4+4x+8+1 \\ &= -x^2+5 \end{aligned}$$

For You Information

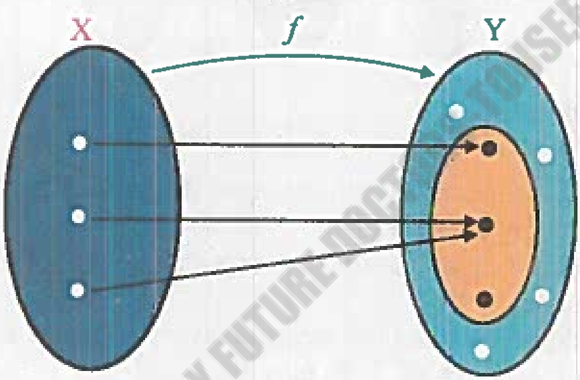
Although f is often used as a convenient function name and x is often used as the independent variable, other letters can also be used. For example, $f(x)=x^2-7x+12$, $f(t)=t^2-7t+12$, and $g(s)=s^2-7s+12$ all define the same function.

8.1.2 Domain and Range of a Function

Let $f: X \to Y$ be a function from a set X to a set Y. Then set X is called **domain** and the set Y is called **codomain** of the function f. The set of all those elements of Y which f is assuming is called **range** of the function f.

If the domain is not specified, then it is assumed to be the set of all real numbers.

If f is a function of X into Y, the range is a subset of Y but need not be all of Y. This has been shown in (Figure 8.4).



Domain Figure (8.4) *Range* $\subseteq$ codomain

Did You Know

Function Notation

$y=f(x)$ f is the *name* of the function.

y is the dependent variable.

x is the independent variable.

y is the *value of the function at x*

8.1.3 One-to-one and onto Function

(a) A function $f: X \to Y$ is said to be **one-to-one** (or **injective**) if distinct elements of X have distinct images in Y i.e. if x_1 and x_2 are distinct elements of X, then $f(x_1) \neq f(x_2)$ in Y. Equivalently, if $f(x_1)=f(x_2)$, then $x_1=x_2$.

Sometimes we write 1-1 function for one-to-one function.

(b) A function $f: X \to Y$ is said to be **onto** (or **surjective**) if each element of Y is the image of some element in X i.e. the range of f is the whole set Y.

A function f which is both one – to – one and onto is called bijective function. Consider the functions f and g as shown in [Figure (8.5) (i) and (ii)].

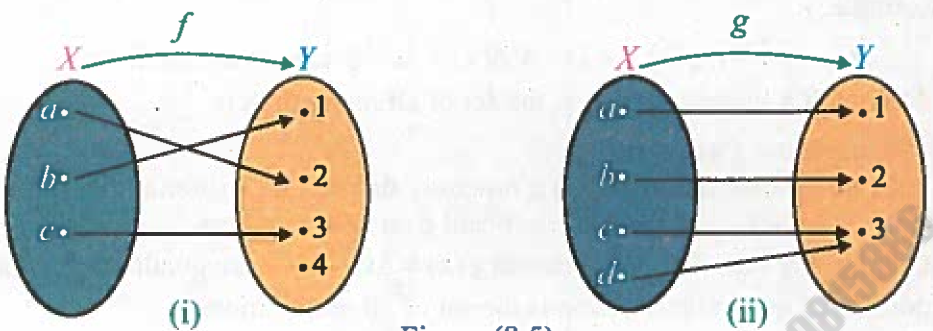


Figure (8.5)

Figure (i) represents a function *f* which is one –to – one but not onto (why?)
Figure (ii) represents a function g which is onto but not one-to-one (why?)

Example 3: Show that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3 - 5x$ is both one-to-one and onto i.e. bijective.

Solution: For any two elements x_1 and x_2 of X, we have

$$f(x_1) = 3 - 5x_1 \text{ and } f(x_2) = 3 - 5x_2$$

If $f(x_1) = f(x_2)$, then $3 - 5x_1 = 3 - 5x_2 \Rightarrow x_1 = x_2$.

Thus f is one-to-one.

Now the range of $f(x) = 3 - 5x$ is the whole set $\mathbb{R}$ so it is onto.

Hence *f* is both one-to-one and onto i.e bijective.

Example 4: Show that the function $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x^2 + 1$ is neither one-to-one nor onto.

Solution: The function $g(x) = 2x^2 + 1$ is not one-to-one, because $g(-2) = 2(-2)^2 + 1 = 9 = 2(2)^2 + 1 = g(2)$, that is – 2 and 2 both have the same image 9.

Now the range of $g(x) = 2x^2 + 1$ is the set of real numbers greater than or equal to 1, that is, Range $g = [1, \infty) \neq \mathbb{R}$, so g is not onto function. Thus g is neither one-to-one nor onto.

8.1.4 Linear, Quadratic and Square Root Functions

We begin with the definition of:

(a) Linear Functions

A function *f* is a **linear function** if it can be written as $f(x) = mx + b$, where m and b are constants.

(If m =0, the function is a **constant function** $f(x) = b$, if m =1 and b= 0, the function is the **identity function** $f(x) = x$)
For example,

$$f(x)=x+1, g(x)=-3x+4, h(x)=3x-8 \text{ are linear functions.}$$

The domain of a linear function is the set of all real numbers.

(b) Quadratic Functions

A **quadratic function** f is a function that can be written in the form $f(x)= ax^2 + bx + c,\ a \neq 0$, where a, b and c are real numbers.
For example, $f(x)=3x^2+4x+1$, and $g(x)=5x^2-x-7$ are quadratic functions.
The domain of quadratic function is the set of all real numbers.

(c) Square Root Function

A function of the form $f(x)=\sqrt{x}$ where $x \geq 0$, is called a square root function.
The domain of square root function is the set of all non-negative real numbers.

8.2 Inverse Function

Let $f: X \to Y$ be a one-to-one and onto function. Then for each element in the domain of f, there is a unique element in the range of f and for each element in the range of f, there is a unique element in the domain of f. In this case the correspondence $f^{-1}: Y \to X$ is also a function, which is called an inverse function of f. Thus the inverse function f^{-1} of f is defined by

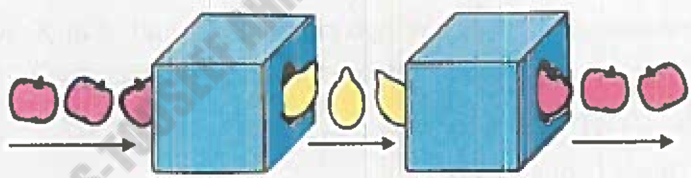


Figure (8.6)

$$x = f^{-1}(y), \forall y \in Y \text{ if and only if } y = f(x), \forall x \in X$$

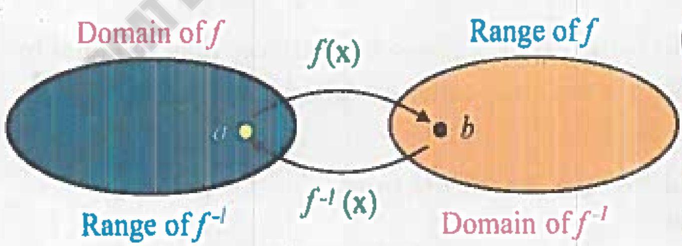


Figure (8.7)

Remember

(i) Not every function has an inverse
(ii) A function has an inverse if and only if it is 1-1 and onto

It is evident that $(f^{-1})^{-1} = f$. Thus f and f^{-1} are inverses of each other.
The above figure illustrates the concept of inverse function.

8.2.1 Domain and Range of Inverse Functions

It is clear from the definition of inverse function

f^{-1} that *domain* f^{-1} = *range* f *and range* f^{-1} = *domain* f

Example 5: If $f : X \to Y$ is given by

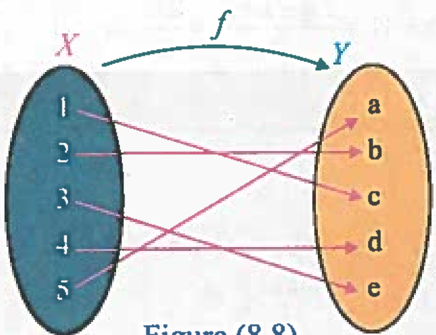


Figure (8.8)

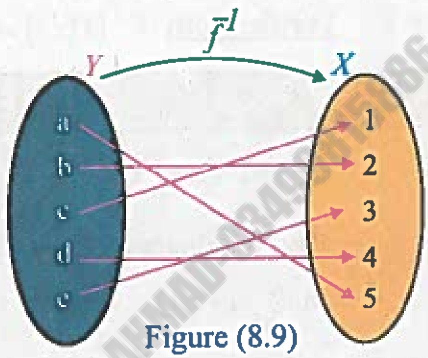


Figure (8.9)

Find f^{-1}.

Solution: Since f is both one-to-one and onto, so it's inverse exists, shown in the (Figure 8.9). We note that f^{-1} is also bijective.

Algebraic method for finding the inverse of a function

If the function f is given by a simple formula, then the inverse function f^{-1} can be found by an algebraic method which involves the following steps.

Step-I Write $y = f(x)$

Step-II Solve the equation in step-I for x in terms of y.

Step-III In the resulting equation in step-II, replace x by $f^{-1}(y)$.

Step-IV Replace each y in the result of step-III by x to get $f^{-1}(x)$

Step-V Check the answer by verifying that $f^{-1}(f(x)) = x$.

Example 6: Let f = $\mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 2x - 1$, find $f^{-1}(x)$.

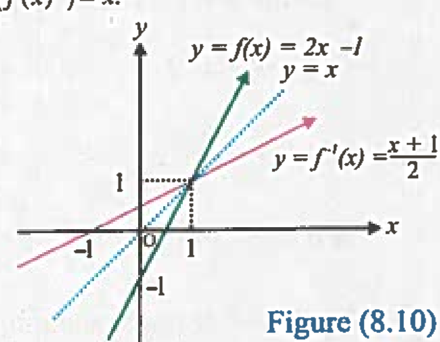


Figure (8.10)

Solution: We have $f(x) = 2x - 1$

Step-I Write $f(x) = 2x - 1 = y$

Step-II Then $2x - 1 = y$

$\Rightarrow \quad 2x = y + 1 \Rightarrow \quad x = \dfrac{y+1}{2}$

Step-III. Replace x by $f^{-1}(y)$ so that

$$f^{-1}(y)=\frac{y+1}{2}$$

Step-IV. To find $f^{-1}(x)$, replace y by x, we have

$$f^{-1}(x)=\frac{x+1}{2}$$

Step-V. <u>**Verification:**</u> $f^{-1}(f(x))=f^{-1}(2x-1)=\dfrac{2x-1+1}{2}=\dfrac{2x}{2}=x.$

EXERCISE 8.1

1. $f(x)=x^2+x-1,$

   (i) Find the images $-2, 0, 2, 5$ (ii) If $f(x)=5$, then find the values of x

   (iii) Find $f(x+1)$ (iv) Find $\dfrac{f(x+h)-f(x)}{h}$

2. If $f(x)=7x-2$, $g(x)=\dfrac{2x}{x^2-4}$, $h(x)=4\sqrt{25-x^2}$, $k(x)=x^2+1$, then determine (i) $f(6), g(-1), h(4), k\left(\dfrac{1}{2}\right)$ (ii) $\dfrac{f(x)-f(2)}{x-2}$

3. Find all real values of x such that $f(x)=0$.

   (i) $f(x)=15x-3$ (ii) $f(x)=x^2-8x+15$

   (iii) $f(x)=x^3-x$ (iv) $f(x)=x^3-x^2-5x+5$

4. Find the domain and range of the function f(x).

   (i) $f(x)=5x^2+2x-1$ (ii) $f(x)=\sqrt{x^2-16}$

5. Find the inverse function of the following functions

   (i) $f(x)=2x-3$ (ii) $f(x)=\dfrac{1}{3}x-5$ (iii) $f(x)=\dfrac{2-x}{5}$ (iv) $f(x)=4+\sqrt{2x}$

6. If $f(x)=x^3-2$, find (i) $f^{-1}(x)$ (ii) $f^{-1}(3)$

7. If $f(x)=\dfrac{x-4}{x-3}$

   Find (i) Domain and range of f. (ii) Domain and range of f^{-1}

8.3 Graphical Representation of Functions

This section is devoted to the representation of functions by graph. The **graph of a function** is a pictorial representation of function that is obtained by using the xy-plane.

Let f be a function defined by $y = f(x)$. The set of all points (x, y) such that x is in the domain of f is called the **graph** of f and we say that the point (x, y) is on the graph of f. To be more specific, if G **denotes** the graph of f, then

$G = \{(x, y) : y = f(x)$ where x is in the domain of $f\}$.

Equivalently, the graph of f is the graph of the equation $y = f(x)$.

The graph of a function may be obtained by constructing a table of corresponding values x of f. Each of these points may be plotted by placing a dot at appropriate location in the xy-plane. Then joining them together by means of a smooth curve gives the required graph of the function.

8.3.1(a) Graphs of Linear Functions

We sketch the graph of linear functions of the form $y = ax + b$ where $a, b \in I\!R$ and $a \neq 0$.

Example 7: Sketch the graph of the function

$f(x) = 2x + 1,\ x \in \{0, 1, 2, 3, 4\}$

Solution: For graph of this function, we assign values to x from its domain and find the corresponding values of y in the range of f as shown in the table:

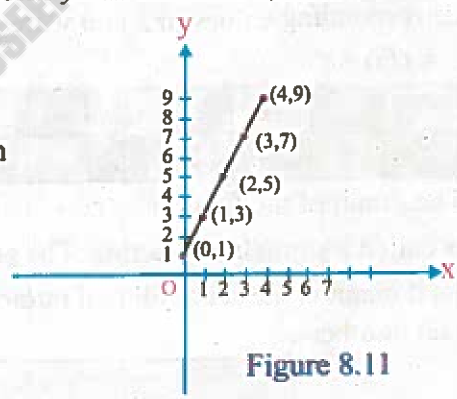


Figure 8.11

$y = f(x) = 2x + 1$

x	0	1	2	3	4
y	1	3	5	7	9

Plotting the points (x, y) in Cartesian plane and joining them with curve, we get graph of the given function as shown in the (Figure 8.11).

> **Note**
> It is clear from the above figure, that the graph of a linear function is a straight line.

Example 8: Draw the graph of the function $y = f(x) = 2x + 1, x \in I\!R$.

Solution: The domain of the function is the set of all real numbers $I\!R$. For the graph of $y = f(x) = 2x + 1$, we assign some values to x from its domain and find corresponding values y in the range of f as shown in the table:

$y = f(x) = 2x + 1$

x	-3	-2	-1	0	1	2	3
y	-5	-3	-1	1	3	5	7

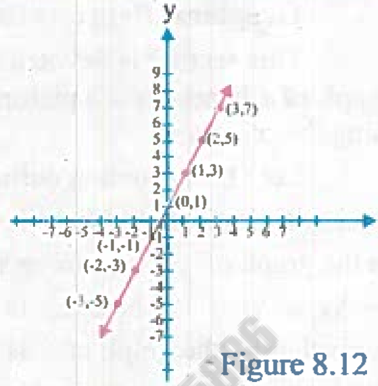


Figure 8.12

The graph of the function is shown in figure (8.12). As x can be any real number, the line is infinite in both the directions representing all the real numbers in the line. The domain and range of linear function are the set of all real numbers.

(b) Graph of Non-linear functions

In this section, we will sketch the graph of non-linear functions, that is functions of the form $f(x) = x^2, f(x) = x^3$ and so on.

Example 9: Sketch the graph of the function $y = f(x) = x^2$

Solution: In the following table some of the corresponding values of x and y are given $y = f(x) = x^2$.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

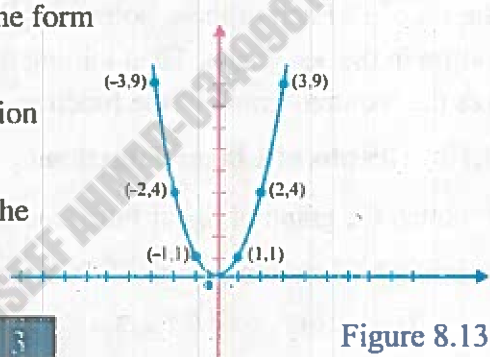
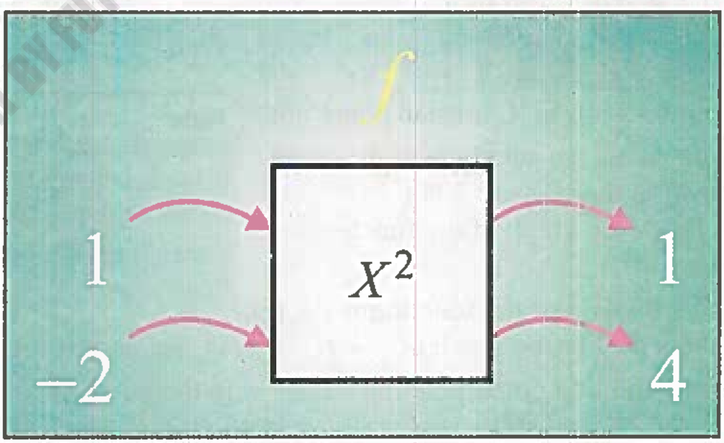


Figure 8.13

The graph of the function $f(x) = x^2$ is shown in figure (8.13). The function $f(x) = x^2$ is called a **squaring function.** The graph of squaring function is called a **parabola.** Its domain is the set of all real numbers and its range is the set of non-negative real numbers.



Example 10: Let $f(x) = x^3$. Sketch the graph of f.

Solution: We construct a table of values for $f(x) = x^3$ as follows:

$y = x^3$

x	-3	-2	-1	0	1	2	3
y	-27	-8	-1	0	1	8	27

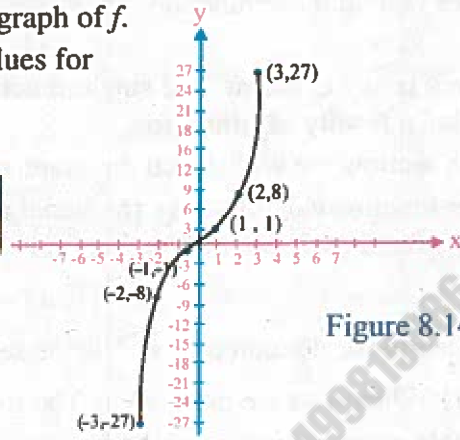


Figure 8.14

Plotting the corresponding points and joining them by a smooth curve, we obtain the graph of the function in figure (8.14). The function $f(x) = x^3$ is called a **cubing function.**

The domain and range of the cubing function are the set of all real numbers.

Example 11: Sketch the graph of the function $f(x) = \sqrt{x}$.

Solution: The given function f is a square root function. The following table gives some values of y corresponding to values of x.

$y = f(x) = \sqrt{x}$

x	0	1	4
y	0	1	2

The graph the function is shown in figure 8.15.

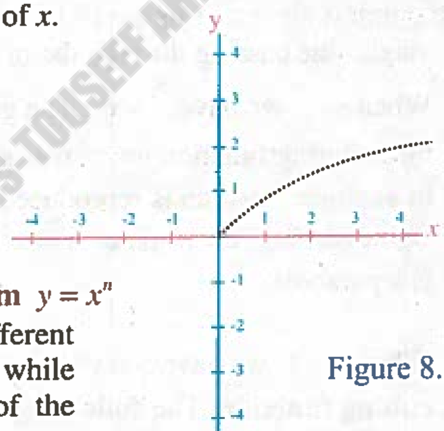


Figure 8.15

8.3.2 Graph of the function of the form $y = x^n$

Sometimes we group together different functions and write them in a single form while observing .the definition and properties of the functions. For example, consider the **power function** $y = x^m$ where m is any constant.

Now, if

(a) $m = n$ i.e. a positive integer, we have another function of the form $y = x^n$

(b) $m = -n$ i.e. a negative integer, we have another function of the form
$y = x^{-n} = \dfrac{1}{x^n}, x \neq 0$

(c) $m = \dfrac{1}{n}$ i.e. a rational number, we have yet another function of the form
$y = x^{\frac{1}{n}}, x > 0$

We see that all these functions are represented by a single function of the form

$$y = x^n \qquad (1)$$

where n is any constant. The single function in (1) representing different functions is called a **family of function.**

In this section, we will sketch the graph of the family of functions $y = x^n$. The power function can also have fractional and irrational exponents. However, the discussion of such power functions is beyond the scope of this book.

(a) Graph of $y = x^n$ where n is a positive integer

Clearly the domain of $y = x^n$ is the set of real numbers.

(1) When $n = 1$, we have $y = x$. The following table gives the values of the function $y = f(x) = x$

x	-2	-1	0	1	2
y	-2	-1	0	1	2

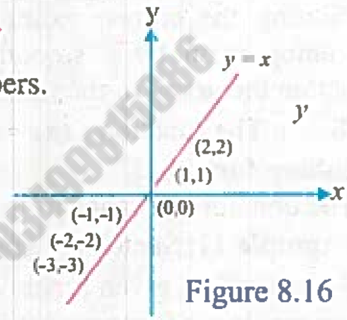


Figure 8.16

The graph is shown in figure (8.16) which is a straight line passing through the origin.

(2) When $n = 2$, we have $y = x^2$. The graph of the squaring function $y = x^2$ was sketched in example 9 which is reproduced in figure (8.17). The graph of $y = x^2$ is a parabola.

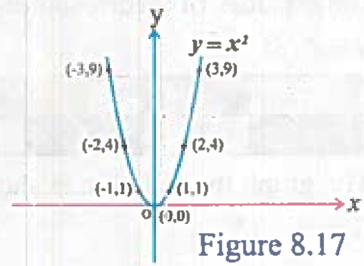


Figure 8.17

(3) When $n = 3$, we have $y = x^3$ which is called **cubing function**. The following table gives some values of the cubing function $y = x^3$.

x	-2	-1	0	1	2
y	-8	-1	0	1	8

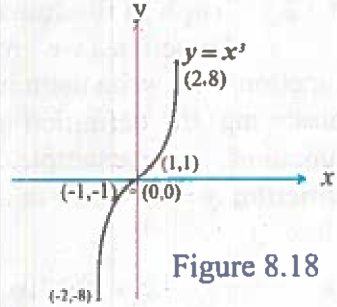


Figure 8.18

The graph of the function is shown in figure (8.18)

(4) When $n = 4$, we have $y = x^4$,

The following table gives some values of the function $y = x^4$

x	-2	-1	0	1	2
y	16	1	0	1	16

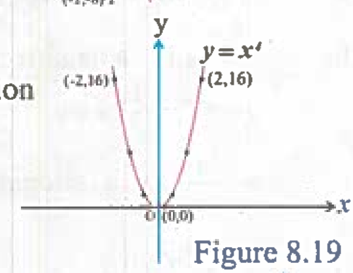


Figure 8.19

The graph is shown in figure (8.19)

(5) When $n = 5$, we have $y = x^5$. The following table gives some values of the function $y = x^5$

x	-2	-1	0	1	2
y	-32	-1	0	1	32

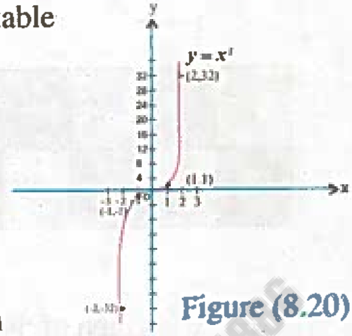


Figure (8.20)

The graph of the function is shown in figure (8.20)

Remember that

(i) When the values of n are even, the function $f(x) = x^n$ are even functions and the graphs of the function $f(x) = x^n$ are symmetric about the $y-$axis. In this case, all the graphs have the same general shape as the parabola $y = x^2$

(ii) When the values of n are odd, the functions $f(x) = x^n$ are odd functions and the graphs of the function $f(x) = x^n$ are symmetric about the origin. In this case, all the graphs have the same general shape as $y = x^3$ for odd n greater than 1.

(iii) By increasing n the graphs in both cases become flatter over the interval $-1 < x < 1$ and steeper over the interval $x > 1$ and $x < -1$ as shown in figure(8.21) and figure (8.22).

$f(x) = x^n$ when n is even

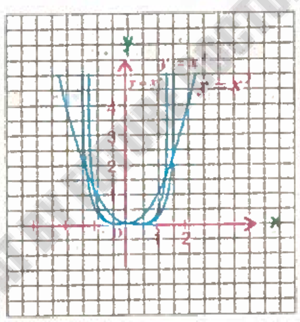


Figure 8.21

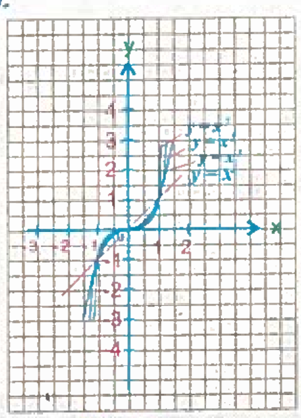


Figure 8.22

$f(x) = x^n$ when n is odd

(b) Graph of $y = x^n$ where n is a negative integer

The domain of the function $y = \frac{1}{x^n}$ is the set of all real numbers except $x \neq 0$.

(1) when $n = -1$ we have $y = \frac{1}{x}$. Some of the values of the function are given in the following table.

$y = \frac{1}{x}$

x	-3	-2	$-\frac{1}{2}$	-1	1	2	$\frac{1}{2}$	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-2	-1	1	$\frac{1}{2}$	2	$\frac{1}{3}$

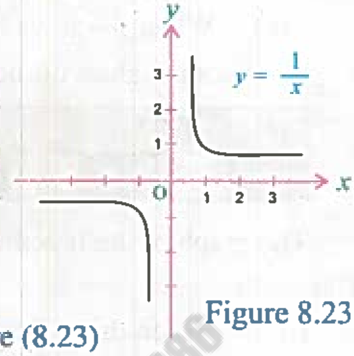


Figure 8.23

The graph of the function is shown in figure (8.23)

(2) When $n = -2$, we have $y = \frac{1}{x^2}$. In the following table some of the values of the function are given.

$y = \frac{1}{x^2}$

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	$\frac{1}{4}$	1	4	4	1	$\frac{1}{4}$

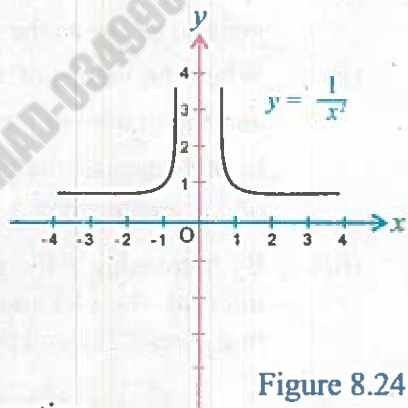


Figure 8.24

Figure (8.24) represents graph of the function.

(3) When $n = -3$, we have $y = \frac{1}{x^3}$. The following tables gives some values of the function.

$y = \frac{1}{x^3}$

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	$-\frac{1}{8}$	-1	-8	8	1	$\frac{1}{8}$

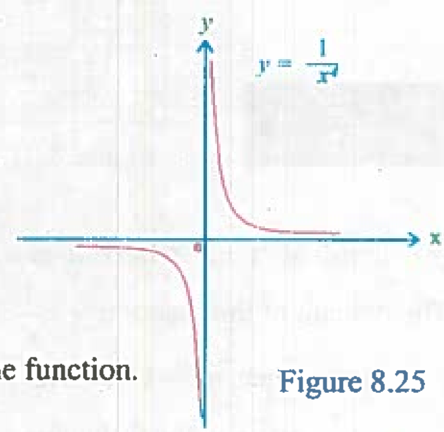


Figure (8.25) shows the graph of the function.

Figure 8.25

(4) When $n=-4$, we have $y=\frac{1}{x^4}$. The following table gives some values of the function.

$y=\frac{1}{x^4}$

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	$\frac{1}{32}$	1	32	32	1	$\frac{1}{32}$

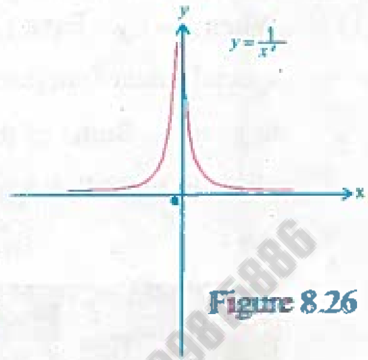


Figure 8.26

The graph of the function is shown in figure (8.26)

Remember that

(i) When the values of n are even, the functions $f(x)=\frac{1}{x^n}$ are even, and their graphs are symmetric about $y-axis$. In this case, all the graphs have the same general shape as $y=\frac{1}{x^2}$.

(i) When the values of n are odd, the functions $f(x)=\frac{1}{x^n}$ are odd, and their graphs are symmetric about the origin. In this case, all the graphs have the same general shape as $y=\frac{1}{x}$.

(ii) By increasing n, the graphs in both cases become steeper over the intervals $-1<x<0$ and $0<x<1$, and flatter over the intervals $x>1$ and $x<-1$ as shown in figure (8.27) and figure (8.28) respectively.

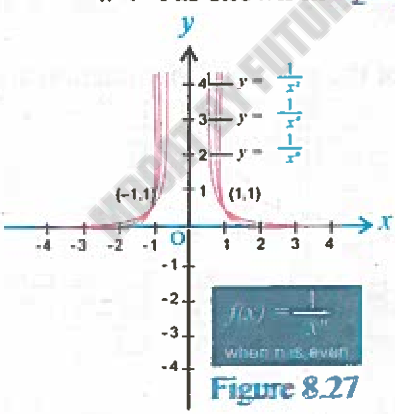


Figure 8.27

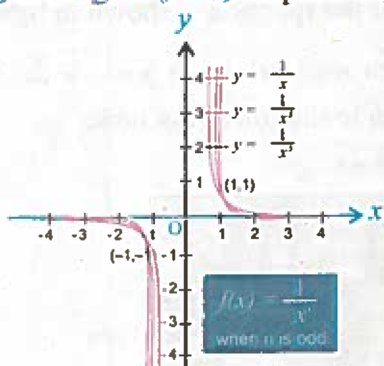


Figure 8.28

(c) Graph of $y=x^n\ (x>0)$ when n is a Rational Number

Generally the domain of the function $y=x^{\frac{1}{n}}$ is the set of all real numbers.

However, at present we will consider $y = x^{\frac{1}{n}}$ with $x > 0$.

(1) When $n = 1$, we have $y = f(x) = x$ which is the **identity function**. It is a special linear function. Its domain and range are the set of all real numbers in general. Some of the values of the function are given in the following table.

$y = x$

x	1	2	3	4	5	6
y	1	2	3	4	5	6

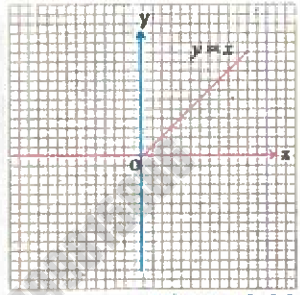


Figure 8.29

The graph of the function is shown in figure (8.29) which is a straight line.

(2) When $n = 2$, we obtain $y = x^{\frac{1}{2}} = \sqrt{x}$ that is, the square root function. The following table gives the values of the function $y = \sqrt{x}$

X	$\frac{1}{4}$	1	4	9	16
y	$\frac{1}{2}$	1	2	3	4

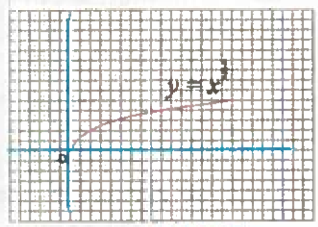


Figure 8.30

The graph of the function is shown in figure(8.30)

(3) When $n = 3$, we have $y = x^{\frac{1}{3}} = \sqrt[3]{x}$. Some of the values of the function are given in the following table.

$y = \sqrt[3]{x}$

x	$\frac{1}{8}$	1	8
y	$\frac{1}{2}$	1	2

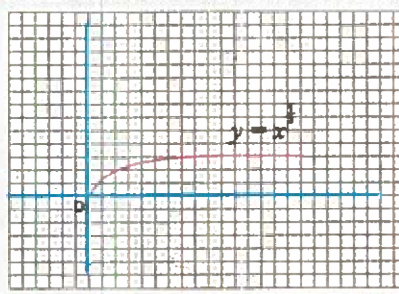


Figure 8.31

The graph of the function is shown in figure(8.31).

(4) When $x = 4$, we have $y = x^{\frac{1}{4}} = \sqrt[4]{x}$. The values ofthe function are given in the following table

$y = \sqrt[4]{x}$

x	$\frac{1}{16}$	1	16
y	$\frac{1}{2}$	1	2

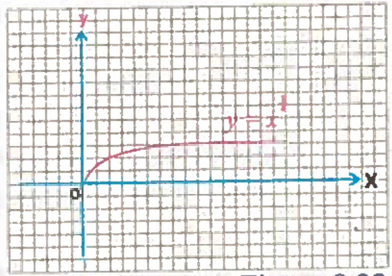


Figure 8.32

The graph is given in figure(8.32).

Remember that

(i) When the values of n are even, the graphs of the function $y = x^{\frac{1}{n}}$ have the same general shape as the square root function $y = \sqrt{x}$

(ii) When the values of n are odd, the graphs of the functions $y = x^{\frac{1}{n}}$ have the same general shapes as $y = x^{\frac{1}{3}} = \sqrt[3]{x}$.

(iii) The graph of $y = x^{\frac{1}{3}}$ extends over the entire $x-axis$, because $f(x) = x^{\frac{1}{3}}$ is defined for all real values of x. The reason is that every real number has a cube root.

(iv) The graph of $y = x^{\frac{1}{2}}$ only extends over the non–negative $x-axis$. The reason is that negative numbers have imaginary roots.

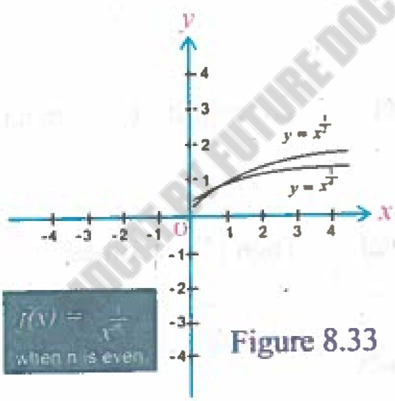


Figure 8.33

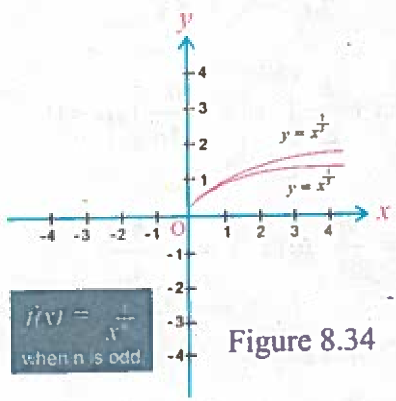


Figure 8.34

8.3.3 The Graph of Quadratic Functions

In this section we want to look at the graph of a quadratic function. The most general form of a quadratic function is, $f(x) = ax^2 + bx + c$

The graphs of quadratic functions are called **parabolas.**

Here are some examples of parabolas

The lowest or highest point of a parabola is called its **vertex.** The vertical line passing through the vertex of a parabola is called the **axis of symmetry** or more briefly **axis** of the parabola.

In figure (8.35), the dashed line passing through the lowest or highest point i.e. vertex of the parabola is the axis of symmetry.

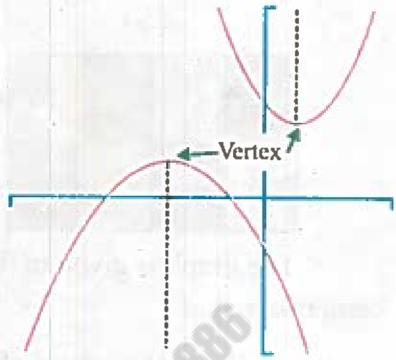


Figure 8.35

The Graph of a General Quadratic Function

Let $f(x)=ax^2+bx+c, a\neq 0$ be an arbitrary quadratic function. In order to sketch graph, we complete the square in $f(x)=ax^2+bx+c$ as follows:

$$f(x)=ax^2+bx+c$$

$$=(ax^2+bx)+c \qquad \text{(Separating c)}$$

$$=a(x^2+\frac{b}{a}x)+c \qquad \text{(Taking a as common)}$$

$$=a(x^2+\frac{b}{a}x+\frac{b^2}{4a^2})+c-a(\frac{b^2}{4a^2}) \qquad \text{(Adding and subtracting the square of half of the co - efficient of } x\text{)}$$

$$=a(x+\frac{b}{2a})^2+(c-\frac{b^2}{4a})$$

$$f(x)=a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right), a\neq 0 \qquad (1)$$

To simplify (1), we let

$$h=\frac{-b}{2a} \text{ and } k=c-\frac{b^2}{4a} \qquad (2)$$

Then (1) becomes,

$$f(x)=a(x-h)^2+k \qquad (3)$$

The graph of f is a Parabola with vertex at the point (h,k)

The parabola opens upward if $a>0$ and downwards if $a<0$

The axis is the vertical line $x=h$. With the help of formula (3), we can draw a reasonably accurate graph of the quadratic function in x by plotting the vertex and at least two points in each side of it.

Example 12: Sketch the graphs of the quadratic functions f and g defined by

(a) $f(x)=x^2$ (b) $g(x)=-x^2$

Solution:(a) The graph of the quadratic function $f(x)=x^2$ with $a=1,\ b=0,\ c=0$ was sketched in Example 9 and is reproduced in figure (8.36).

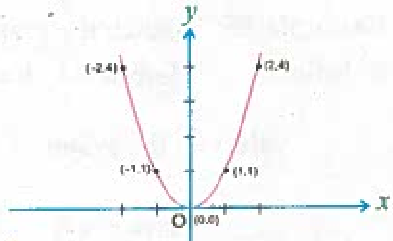


Figure 8.36

The vertex of the graph is the lowest point (0,0).

(b) In the following table some of the values of x and corresponding values of y of the quadratic equation $y=g(x)=-x^2$ with $a=1,\ b=0,\ c=0$ are given:

$y=g(x)=-x^2$

x	-2	-1	0	1	2
y	-4	-1	0	-1	-4

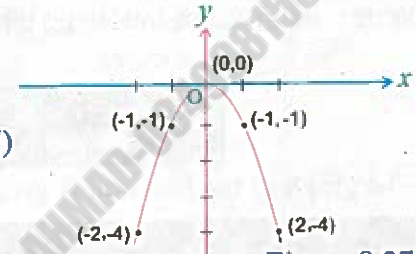


Figure 8.37

The graph of the function is shown in figure (8.37)

The graph of $f(x)=x^2$ **opens upward** and the graph of $y=g(x)=-x^2$ **opens downward.**

In general if, $f(x)=ax^2, a\neq 0$, then the graph of $f(x)$ opens upward if $a>0$ and opens downward if $a<0$

Example 13: Sketch the graph of the function

$$f(x)=x^2-2x+1$$

Solution: We construct a table of values of the function as follows: $y=x^2-2x+1$

x	-3	-2	-1	0	1	2	3	4	5
y	16	9	4	1	0	1	4	9	16

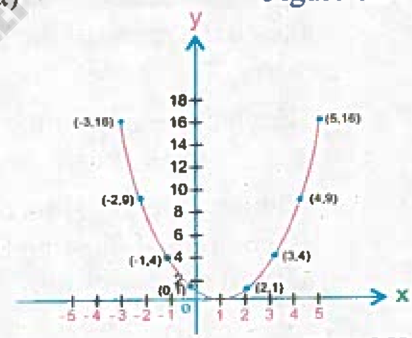


Figure 8.38

The graph of the function is shown in figure (8.38) with vertex at (1,0)

Example 14: Without graphing, find the vertex and axis of the graph of the function $f(x)=-x^2+4x-5$. Also determine whether the graph opens upward or downward.

Solution: Here $a=-1,\ b=4,\ c=-5$.

$\therefore$ vertex of the graph of $f=(h,k)=(-\frac{b}{2a},\ c-\frac{b^2}{4a})$

$$=(-\frac{4}{2(-1)},-5-\frac{(4)^2}{4(-1)})=(2,-1).$$

$$\text{Axis}=x=-\frac{b}{2a}=2$$

Since $a=-1<0$, so the graph opens downward.

Example 15: Sketch the graph of the function $f(x)=x^2-2x-2$.

Solution: Here $a=1,\ b=-2,\ c=-2$.

$\therefore$ vertex of the graph of $f=\left(-\dfrac{b}{2a},\ c-\dfrac{b^2}{4a}\right)=\left(-\dfrac{-2}{2(1)},-2-\dfrac{(-2)^2}{4(1)}\right)=(1,-3)$

$$\text{Axis} = x=-\frac{b}{2a}=1$$

Since $a=1>0$, so the graph opens upward.

The two additional values on each side of the vertex are given in following table.

$y=x^2-2x-2$

x	-1	0	1	2	3
y	1	-2	-3	-2	1

The graph of the function is given in figure (8.39).

Figure 8.39

EXERCISE 8.2

1. Sketch the graph of the given function

   (i) $f(x)=2x+3$ (ii) $f(x)=4x-5$

2. Sketch the graphs of the following functions

   (i) $f(x)=x^2+1$ (ii) $f(x)=-x^2+1$ (iii) $f(x)=x^2+2x+1$

3. Without graphing, find the vertex, all intercepts if any and axis of the graph of the following function. Also determine whether the graphs ope upward or downward.

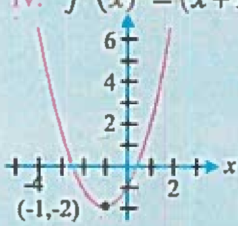
   (i) $f(x)=\dfrac{3}{4}x^2$ (ii) $f(x)=-2x^2+8$

   (iii) $f(x)=-x^2+6x-5$ (iv) $f(x)=x^2+2x-3$

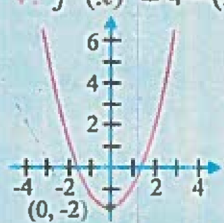
4. Match the quadratic function with its graph. [The graphs are labeled (a), (b),(c), (d), (e), and (f).]

   i. $f(x)=(x-2)^2$ ii. $f(x)=(x+4)^2$ iii. $f(x)=x^2-2$

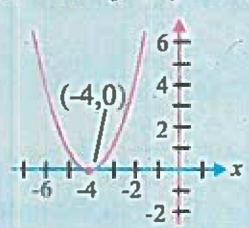
   iv. $f(x)=(x+1)^2-2$ v. $f(x)=4-(x-2)^2$ vi. $f(x)=-(x-4)^2$



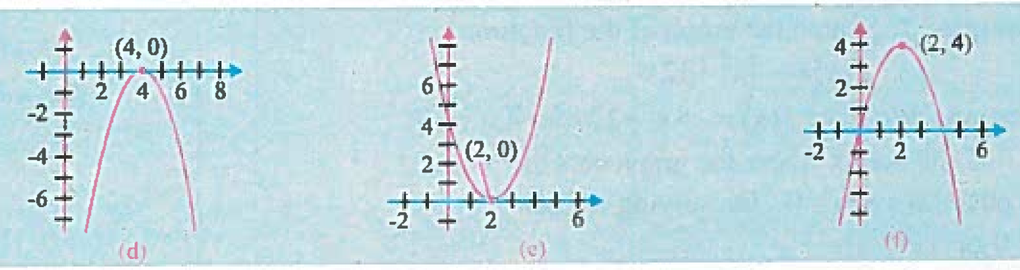
(a)



(b)



(c)



8.3.4 Using Factors to Sketch Graphs

In the above section we sketched the graphs of quadratic functions by plotting many points. In this section too, we will sketch the graphs of quadratic functions but using their factors.

We know from our previous class knowledge that a quadratic expression can be written as a product of factors. For example, we can write

$$x^2+3x+2=(x+1)(x+2)$$

where $(x+1)$ and $(x+2)$ are the factors of the quadratic expression x^2+3x+2. Similarly, some quadratic functions of the form $f(x)=ax^2+bx+c\ (a\neq 0)$ can be factored and their graphs can be drawn by using the factors. This method of using factors to sketch the graph of quadratic function is explained through the following examples.

Example 16: Sketch the graph of the function $f(x)=x^2+2x-3$.

Solution: We have $f(x)=2x^2+2x-3=(x+3)(x-1)$.

To find the points which lie on the graph of the function $f(x)$,

we put $(x+3)(x-1)=0$. The equation is satisfied if $x=-3$ *or* $x=1$.

Now $f(-3)=0$ and $f(1)=0$. Thus the points lying on the graph of $f(x)$ are $(-3,0)$ and $(1,0)$ that is, the graph cuts the x–axis at $(-3,0)$ and $(1,0)$.

To find the point where the graph cuts the $y-axis$ we put $x=0$ in the function so that $f(0)=-3$. Therefore the required points is $(0,-3)$. All that remains to be done is to obtain few additional points on the graphs in order to sketch it. Some of these are given in the table below.

$y=(x+3)\ (x-1)$

x	-4	-2	-1	0	2
y	5	-3	-4	3	5

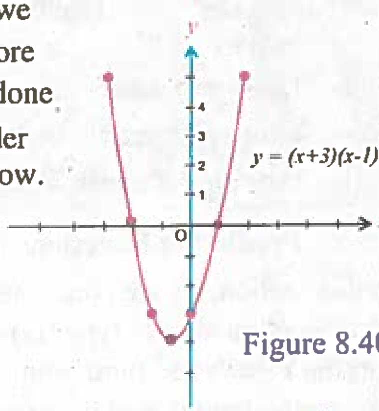


Figure 8.40

The graph of the function is shown in figure (8.40), which opens upward, since $a=1>0$.

Example 17: Sketch the graph of the function

$$f(x) = -4x^2 + 12x\ .$$

Solution: We have $f(x) = -4x^2 + 12x = -4x(x-3)$

To find the points where the graph cuts the $x-axis$, we put $-4x(x-3) = 0$. On solving we get

$x = 0$ or $x = 3$.

$\therefore f(0) = 0$ and $f(3) = 0$

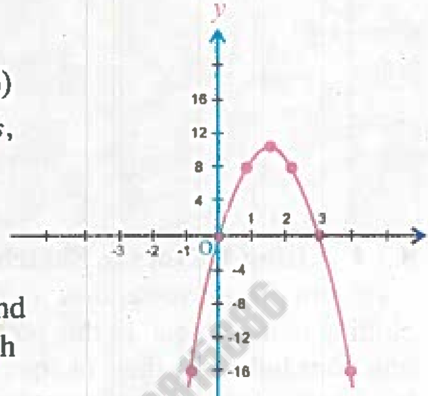


Figure 8.41

Thus the graph cuts $x-axis$ at the points (0,0) and (3,0). Also $f(0) = 0$, so the point where the graph cuts $y-axis$ is (0,0).

To draw the graph, we need some additional points, which are given in the table below:

$$y = -4x\ \ (x-3)$$

x	-1	1	$\frac{3}{2}$	2	4
y	-16	8	9	8	-16

The graph of the function is given in figure (8.41) which opens downward, since $a = -4 < 0$.

Remember

We may draw the graph of any quadratic function $f(x)$ which can be factorized as $y = f(x) = a\,(x-p)\,(x-q)$ by keeping the following points in mind.

(i) Note the points $(p,0)$ and $(q,0)$ where the graph of the function cuts the *x-axis*.

(ii) By taking $x = 0$ in the function $f(x)$, note the point $(0, y)$ where the graph cuts the $y-axis$

(iii) The sign of the constant a tells whether the graph opens upwards or downwards.

(iv) To draw the graph, obtain some additional points on the graph.

(v) The shape of graphs of all quadratic functions is a parabola.

8.3.5 Predicting Functions from their Graphs

In this section, we are concerned with the use of factor form to predict the equation of a function of the type $f(x) = ax^2 + bx + c$, $(a \neq 0)$ if two points where the graph cuts the $x-axis$ and third point on the curve are given.

The method employed in doing so is explained through the following example.

Example 18: Find the equation of the graph of the function of the type $y = ax^2 + bx + c$, $(a \neq 0)$ which cuts the $x-axis$ at the point $(-2,0)$ and $(2,0)$ and also passes through the point $(1,-6)$.

Solution: The equation of the curve which passes through $x-axis$ at the points $(p,0)$ and $(q,0)$ has the form $y = a(x-p)(x-q)$ (1)

The curve which passes through the points $(-2,0)$ and $(2,0)$ is shown in figure (8.42).

Here $p=-2$, $q=2$, so by (1), we have

$$y = a(x+2)(x-2) \quad (2)$$

The point $(1,-6)$ lies on the curve, so it must satisfy equation (2) and so $-6 = a(1+2)\ (1-2)$

$$\Rightarrow -6 = -3a \Rightarrow a = 2$$

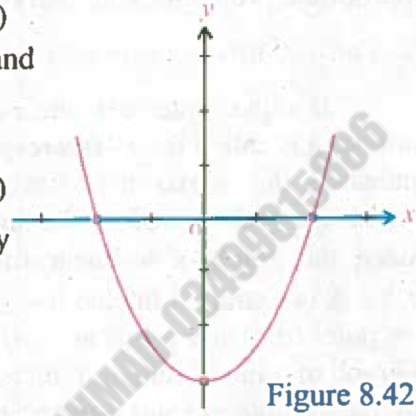


Figure 8.42

Therefore equation (2) of the curve becomes

$y = 2(x+2)(x-2)$ or $y = 2x^2 - 8$, which is the required equation.

Example 19: Find the equation in the form $x^2 + bx + c = 0$ of the parabola which crosses the $x-axis$ at the point $(-5,0)$ and $(3,0)$

Solution: The form of the parabola is given by

$$x^2 + bx + c = 0 \quad (1)$$

The general form of the parabola is given by

$$ax^2 + bx + c = 0 \quad (2)$$

Comparing (1) and (2), we have

$$a = 1 > 0$$

so the parabola opens upward. The equation of the curve which cuts the $x-axis$ at the points (p,o) and (q,o) has the form

$$y = a(x-p)(x-q) \quad (3)$$

but $a = 1$, so (3) becomes

$$y = (x-p)(x-q) \quad (4)$$

The curve which cuts the $x-axis$ at the points $(-5,0)$ and $(3,0)$ is shown in figure (8.43).

We have $p = -5$ *and* $q = 3$

Using (4), we obtain

$y = (x+5)(x-3)$

or $y = x^2 + 2x - 15$

which is the required equation.

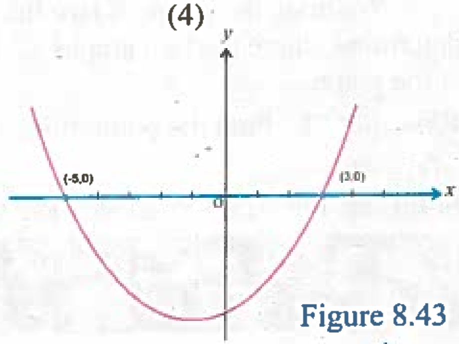


Figure 8.43

8.4 Intersecting Graphs

In this section we aim at to find the intersecting points graphically, when the intersection occurs between a linear function and coordinate axis, two linear functions and a linear and quadratic function. we will also solve graphically appropriate problems from daily life.

(a) Point of intersection of a linear function and coordinate axes

If a line l intersects the *x-axis* at a point $(a, 0)$, the number a is called the ***x- intercept*** of the line l. if a line l intersects the *y-axis* at a point) $(0,b)$ the number b is called the ***y-intercept*** of the line l. see (figure 8.44). Since the graph of a linear function $f(x) = ax + b$, $a, b \in \mathbb{R}$ is a straight line, so it will intersect the *x-axis* at the point $(a,0)$, and *y-axis* at $(0,b)$ thus, the points where a graph of a linear function intersects the coordinate axes are the *x- intercept* and *y-intercept* of the graph.

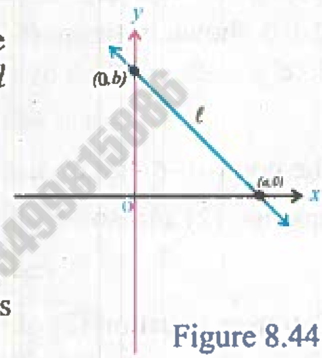


Figure 8.44

Example 20: Find the points of intersection of the linear function $f(x) = x - 4$ with coordinate axes

Solution: By giving some values to x, we find the corresponding values of y in the following table.

$f(x) = y = x - 4$

x	−2	−1	0	1	2	3	4	5	6
y	−6	−5	−4	−3	−2	−1	0	1	2

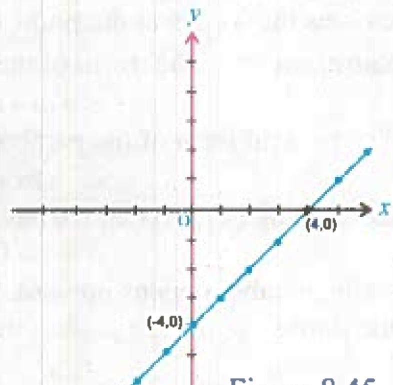


Figure 8.45

The graph of the function is shown in (figure 8.45). The graph intersects *x-axis* at $x = 4$ and *y-axis* at $y = -4$. The answer may be easily verified by finding the *x- intercept* and *y-intercept* of the graph.

(b) Point of Intersection of two linear functions

We draw the graph of two linear functions on the same graph paper and then determine where the two graphs of these two linear functions intersect by looking at the graph.

Example 21: Find the point of intersection of the functions $f(x) = x + 3$ and $g(x) = -2x + 9$.

Solution: For $f(x) = x + 3$, we have the following table of values: $y = x + 3$

x	−5	−4	−3	−2	−1	0	1	2
y	−2	−1	0	1	2	3	4	5

For $g(x) = -2x + 9$, we have the following table of values: $y = -2x + 9$

x	−1	0	1	2	3	4	5
y	11	9	7	5	3	1	−1

The graphs of both functions are shown in (figure 8.46). Looking at the graph, we find that the point of intersection is *(2,5)*

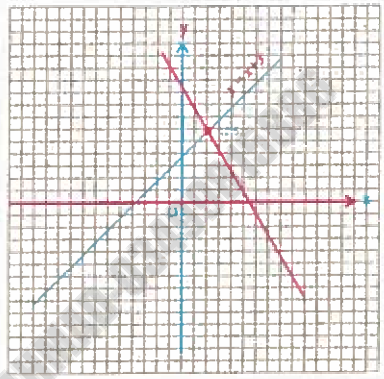


Figure 8.46

Although this seems to be a very simple method of finding the coordinates of the point of intersection of two linear functions, it may not always be very accurate in cases when the coordinates of the point are fractional numbers. In that case, to find where exactly the graphs cross, we must use algebraic rather than graphic method. We can find a value of x and value of y that satisfy both the equations of linear functions simultaneously. For this purpose several methods are available. For example, we may use the method of elimination or method of substitution with whom we are already familiar.

(c) Point of intersection of a linear function and a quadratic function

The method for finding the point of intersection of graphs of a linear function and a quadratic function is the same as that for finding the point of intersection of two graphs of linear functions. The method will be clarified by the following example.

Example 22: Find the point of intersection of the functions $f(x) = x^2 - 4x + 6$ and $g(x) = 2x + 1$

Solution: The following table gives the values of the function $y = f(x) = x^2 - 4x + 6$

$y = x^2 - 4x + 6$

x	−2	−1	0	1	2	3	4	5
y	18	11	6	3	2	3	6	11

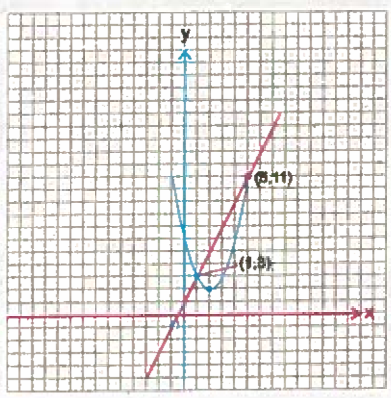


Figure 8.47

The table for values of the function $g(x) = 2x + 1$ is given below: $y = g(x) = 2x + 1$

x	−3	−2	−1	0	1	2	3	4	5	6
y	−5	−3	−1	1	3	5	7	9	11	13

The graphs of these two functions are shown in figure (8.47).
The points of intersection of the two graphs are *(1,3)* and *(5,11)*.

8.4.2 Graphical Solutions of Problems from Daily Life

Many problems from daily life can be solved by means of graphs. Here are some examples.

Example 23: It takes a swimmer 2 min to swim 24m downstream in a river and 4 min to swim back. Find the speed of flow of water and the speed at which he can swim in still water.

Solution: Let x = speed of swimmer in still water and y = speed of flow of water
Therefore speed downstream $= x+y$ and speed upstream $= x-y$
We know that time x speed = distance

$$\therefore \quad 2(x+y)=24$$
$$4(x-y)=24$$
$$\text{or} \quad x+y=12$$
$$x-y=6$$
$$\text{or} \quad y=-x+12 \quad (1)$$
$$y=x+6 \quad (2)$$

we see that equations (1) and (2) are the equations of linear functions and they are represented graphically by straight lines. We find the point of intersection of their graphs.

The values of functions (1) and (2) are given in the following tables: $y = x + 12$

x	-2	-1	0	1	2	6	12	13	14
y	14	13	12	11	10	6	0	-1	-2

and $y = x + 16$

x	-8	-6	-4	-2	0	2	4
y	-2	0	2	4	6	8	12

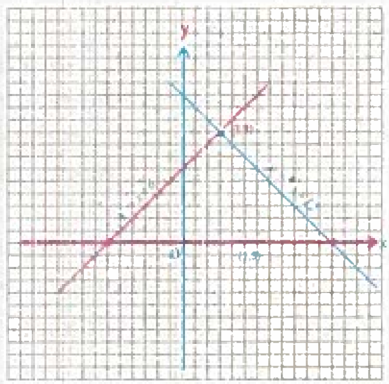


Figure 8.48

The graphs of both functions are shown is (figure 8.48).

We find that their point of intersection is (3,9), that is $x=3$ and $y=9$

Thus the speed of swimmer in still water $= x = 3m/\text{min}$ and the speed of flow of water $= y = 9m/\text{min}$. Use algebraic methods to verify the answer.

Example 24: A group of 45 school children visited a zoo and paid Rs.60.00 altogether as entry ticket. The entry ticket of class 1 was Rs.2.00 per child where as that of class KG Rs.1.00 per child. Find how many children were in the group from each class.

Solution: Let x = the number of children from class 1
and y = the number of children from Class KG.

According to the condition of the question, we have

$$x+y=45$$

$$2x+y=60$$

or $\quad y=45-x \qquad (1)$

$\quad y=60-2x \qquad (2).$

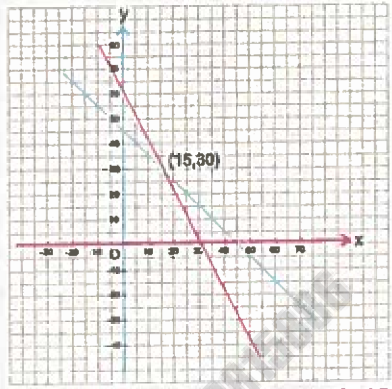


Figure 8.49

Equation (1) and (2) represent the equation of linear functions whose graph are straight line. We find that point of intersection of their graphs. The values of the functions (1) and (2) are given in the following tables.

$y=45-x$

x	−20	−10	0	10	20	30	40	50	60
y	65	55	45	35	25	15	5	−5	−15

and $y=60-2x$

x	−10	0	10	20	30	40	50
y	80	60	40	20	0	−20	−40

The graphs of both functions are shown in figure(8.49). The point of intersection of the graphs is (15,30), that is $x=15$ and $y=30$.

Thus the number of children from class $1=x=15$

and the number of children from class KG. $=y=30$.

EXERCISE 8.3

1. Sketch graphs of the following functions
   (i) $f(x)=(x-1)(x-3)$ (ii) $f(x)=-2(x+1)(x-1)$
2. Using factors to sketch the graphs of the following functions
   (i) $f(x)=x^2-2x-3$ (ii) $f(x)=-(x^2-x-2)$
3. Find the equation of the graph of the function of the type $y=x^2+bx+c$ which crosses the $x-axis$ at the point (3,0) and (4,0).
4. Find the equation of the graph of the function of the type $y=ax^2+bx+c$ which (i) cross the $x-axis$ at the point (–5,0) and (3,0) and also passes through (–1,8)
   (ii) cross the $x-axis$ at the point (–7,0) and (10,0) and also passes through (4,11).
5. Find the point of intersection graphically of the following linear functions with the coordinate axes. (i) $f(x)=x-3$ (ii) $f(x)=2x+1$

6. Find the point of intersection graphically of the following functions.
   (i) $f(x) = -x+2,$ $g(x) = 2x+1$
   (ii) $f(x) = 3x-2,$ $g(x) = -x+6$
7. Find the point of intersection graphically of the following functions.
   (i) $f(x) = -x^2+4,$ $g(x) = x+2$
   (ii) $f(x) = x^2+x-3,$ $g(x) = -2x-5$
8. The paths of two airplanes A and B in the plane are determined by the straight lines $2x-y=6$ and $3x+y=4$ respectively. Find the point where the two paths cross each other.
9. A pilot makes a check flight in an air. Going directly into the wind, he covers a distance of 24 km in 6 minutes. Going with the wind, he covers the distance in 4 minutes. Find his air speed and velocity of the wind in km/min.

REVIEW EXERCISE 8

1. Choose the correct option.

   i. What is the domain of f(x) = $\sqrt{\dfrac{2-x}{x+2}}$?

   (a) $[-4,-2)$ (b) $[0, 2]-\{1\}$ (c) $(-2, 2)$ (d) $(-2, 2]$

   ii. $A = \{-1, 0, 1, 2\}$, $B = \{0, 1, 4\}$ and $f: A \to B$ defined by $f(x) = x^2$, then f is

   (a) Only one-one function (b) Only onto function
   (c) bijective (d) not a function

   iii. If $f: R \to R$ defined by $f(x) = 3x-5$, then $f^{-1}(\{-1, -2, 1, 2\}) =$

   (a) $\left\{1, \frac{4}{3}, \frac{7}{3}\right\}$ (b) $\left\{-1, 2, \frac{4}{3}\right\}$ (c) $\left\{1, 2, \frac{4}{3}, \frac{7}{3}\right\}$ (d) $\{1, 2, -1, -2\}$

   iv. If $f(2x+3) = 4x^2+12x+15$, then find the value of $f(3x+2)$ is
   (a) $9x^2-12x+36$ (b) $9x^2+12x+10$
   (c) $9x^2-12x+24$ (d) $9x^2-12x-5$

   v. If $f(x) = x^3 - \frac{1}{x^3}$, *then* $f(x) + f(\frac{1}{x}) =$

   (a) 0 (b) -1 (c) x^3 (d) None of these

vi. If $f(x) = x^2 - 3x + 4$, then find the values of x satisfying the equation $f(x) = f(2x+1)$

(a) 5/3 (b) 2/3 (c) 1 (d) None of these

vii. The domain of $y = \dfrac{x}{\sqrt{x^2 - 3x + 2}}$ *is*

(a) $(\infty, 1)$ (b) $(2, \infty)$ (c) $(\infty, 1) \cup (2, \infty)$ (d) $(-\infty, 1) \cup (2, \infty)$

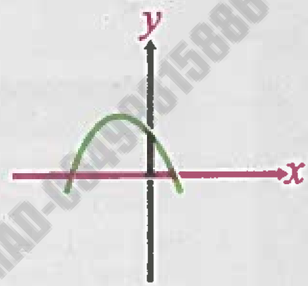
(viii) Guess the quadratic function for the curve given in the figure.

(a) $g(x) = x^2 - 2x - 5$

(b) $g(x) = x^2 + 2x + 5$

(c) $g(x) = -x^2 - 2x + 5$

(d) $g(x) = -x^2 + 2x + 5$



2. Find domain of $f(x) = \sqrt{3 - \sqrt{12 - x^2}}$

3. Find a polynomial function $f(x)$ of the second degree when $f(0) = 5$, $f(-1) = 10$, $f(1) = 6$.

4. Find the range of each of the following functions:

   i) $f(x) = x^2 + 2,\ x \in R$

   ii) $f(x) = x,\ x \in R$

5. The function 't' which maps temperature in Celsius into temperature in degree Fahrenheit is defined by $t(c) = \dfrac{9c}{5} + 32$

   Find (i) $t(0)$ (ii) $t(28)$ (iii) $t(-10)$ (iv) the value of c, when $t(c) = 212$

6. If $f(x) = 8x - 7$, find (i) $f^{-1}(9)$ (ii) $f^{-1}\left(\dfrac{11}{3}\right)$

7. Given that $f(x) = x^3 - ax^2 + bx + 1$. If $f(2) = -3$ and $f(-1) = 0$, find the value of a and b.

8. Graph the following. (i) $y = -\dfrac{1}{2}x + 3$ (ii) $y = -3x^2$ (iii) $y = 2x^2 - 7x + 3$

9. Sketch the graph of the following.

   (i) $y = x^2 + 2x - 3$ (ii) $y = 3(x+1)(x-1)$

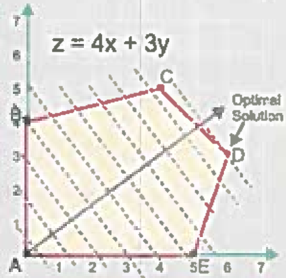
10. Find the point of intersection graphically of the following functions.

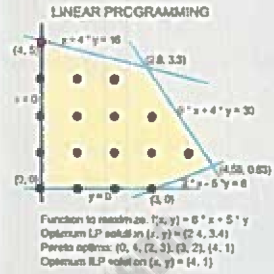
   (i) $f(x) = x + 4,\quad g(x) = -2x + 3$

   (ii) $f(x) = x^2 - x - 2,\quad g(x) = -3x - 3$

UNIT 9

LINEAR PROGRAMMING





STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Define linear programming (LP) as planning of allocation of limited resources to obtain an optimal result.
- Find algebraic solutions of linear inequalities in one variable and represent them on number line.
- Interpret graphically the linear inequalities in two variables.
- Determine graphically the region bounded by up to 3 simultaneous linear inequalities of non-negative variables and shade the region bounded by them.
- Define
    - linear programming problem,
    - objective function,
    - problem constraints,
    - decision variables.
- Define and show graphically the feasible region (or solution space) of an LP problem.
- Identify the feasible region of simple LP problems.
- Define optimal solution of an LP problem.
- Find optimal solution (graphical) through the following systematic procedure:
    - establish the mathematical formulation of LP problem,
    - construct the graph,
    - identify the feasible region,
    - locate the solution points,
    - evaluate the objective function,
    - select the optimal solution,
    - verify the optimal solution by actually substituting values of variables from the feasible region.
- Solve real life simple LP problems.

9.1 Introduction

In business and industry, the decision makers want to utilize the limited resources in a best possible manner with the view to minimize cost of production and maximize profit. The limited resources may be in the form of capital, labour, money, time manpower, machine capacity, etc. The linear programming is the mathematical method used in decision making in business to maximize the profit or minimize the expenditure subject to certain restrictions which are a result of limitations on resources.

The term programming means planning and refers to a process of determining a particular program. The term linear means that all relationships involved in a particular program which can be solved by this method are linear.

Thus linear programming is a method for solving problems in which a linear function (representing, cost, profit, distance, weight etc.) is to be maximized or minimized. Such problems are usually referred to as **optimization problems** or more commonly known as **linear programming problems**.

The theory of linear programming is a fairly recent advancement in mathematics. It was developed over the past four decades to deal with the increasingly more complicated problems of our technological society.

Linear programming (LP) is planning of allocation of limited resources to obtain an optimal result.

9.2 Linear Inequalities

Recall that an inequality is a statement that one mathematical quantity is less than (or greater than) or less than or equal to (or greater than or equal to) another quantity. Thus, if a and b are real numbers, we can compare their positions on the real line by using the relations of less than, greater than, less than or equal to, and greater than or equal to, denoted by inequality symbols $<, >, \leq$ and $\geq$ respectively. The following table describes both algebraic and geometric interpretations of the inequality symbols.

Algebraic Statement	Equivalent Statement	Geometric Statement
$a < b$	a is less than b	a lies to the left of b.
$a > b$	a is greater than b	a lies to the right of b.
$a \leq b$	a is less than or equal to b	a coincides with b or lies to the left of b.
$a \geq b$	a is greater than or equal to b	a coincides with b or lies to the right of b.

In this section, we shall consider linear inequalities in one variable and two variables. We shall also interpret these inequalities graphically.

9.2.1. Linear inequalities in one variable

Inequalities of the form $ax < b$, $ax \leq b$, $ax > b$ or $ax \geq b$ where $a \neq 0$, b are constants are called **linear inequalities in one variable** or **first degree inequalities in one variable.**

For example, $x < -2$, $2x \leq 6$, $4 - 3x > -1 - x$, $2x + 5 \geq x - 3$ are linear inequalities in one variable.

The **solutions** of a linear inequality in one variable x are the values of x which satisfy the linear inequality. The set consisting of all solutions of the linear inequality is called the **solution set.**

For example, the solution set of the linear inequality $x > 5$ consists of all values of x that are greater than 5.

We solve a linear inequality in the same way as we solve a linear equation. Following are the steps involved in solving a linear inequality in one variable.

Step I Shift all terms containing x on one side of the inequality.

Step II Shift all other terms on the other side of the inequality.

Step III Simplify the resulting inequality to find the values of x.

Example 1: Solve the linear inequality $x - 5 > 0$.

Solution: Since the only term containing x is on the left side, we need to shift the constant term to the other side. To do this, we add 5 to both sides and then simplify.

$$x - 5 > 0$$
$$(x - 5) + 5 > 0 + 5$$
$$x > 5$$

Thus, the solution of the inequality are all values of x that are greater than 5.

The solution set = $\{ x : x \in \mathbb{R} \text{ and } x > 5 \}$

The solution set can also be written alternatively in the form of interval $(5, \infty)$.

Example 2: Solve the inequality $3x - 2 \geq 8 + 5x$

Solution: To solve the given linear inequality, we use step (I)–(III) to obtain the following equivalent inequalities.

$$3x - 2 \geq 8 + 5x$$
$$(3x - 2) - 5x \geq (8 + 5x) - 5x$$
$$-2x - 2 \geq 8$$
$$(-2x - 2) + 2 \geq 8 + 2$$

Did You Know ?

When both sides of an inequality are multiplied by a negative number, the order (or sense) of the inequality is reversed, that is from $<$ to $>$, from $\leq$ to $\geq$, from $>$ to $<$ or from $\geq$ to $\leq$.

$-2x \geq 10$

$\left(-\frac{1}{2}\right)(-2x) \leq \left(-\frac{1}{2}\right)(10)$.

$x \leq -5$ Thus, the solution set $= \{ x : x \in \mathbb{R} \text{ and } x \leq -5 \} = (-\infty, -5]$

Note

In the above graphical representation of linear inequalities in one variable on the real line, the open (unshaded) circle at the point indicates that the point does not belong to solution set. The filled in (shaded) circle at the point indicates that the point belongs to the solution set.

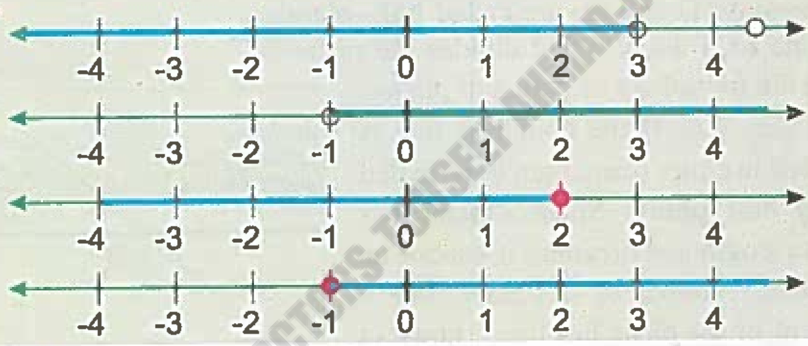
The solutions of linear inequalities in one variable are graphically represented on the real line in the following examples.

(i) $x < 3$

(ii) $x > -1$

(iii) $x \leq 2$

(iv) $x \geq -1$



9.2.2. Linear inequalities in two variables

A linear inequality in two variables x and y is an expression of one of the following forms.

(i) $ax + by < c$ (ii) $ax + by > c$

(iii) $ax + by \leq c$ (iv) $ax + by \geq c$

where a and b are not both 0 and a, b and c are real numbers.

If $a = 0$ or $b = 0$ in the above inequalities, then the resulting inequalities reduce to the corresponding linear inequalities in one variable.

For example, (i) $3x < 2$ (ii) $4x + 3 \geq 0$ (iii) $x - 2y > 1$ (iv) $5x + 3y \leq 1$ are linear inequalities. Inequalities (i) and (ii) are in one variables while (iii) and (iv) are in two variables. With each linear inequality in two variables x and y is associated a linear equation in two variables x and y called the **associated** or **corresponding equation**.

For example, the associated equation of $ax + by \geq c$ is $ax + by = c$........... (1)

To find the associated equation of a linear inequality in two variables, simply substitute an "equals" sign for the symbol of inequality. In our later work we will see that the linear equation (1) in two variables represents a straight line.

The **solution set** of an inequality is the set of all numbers, which when substituted for the variable (or variables) in the inequality, make the inequality a true statement. To solve an inequality is to find its solution set.

9.2.2.1 Graphing Inequalities in Two Variables

Since linear inequalities are closely related to linear equations, graphing them is very similar to graphing linear equations. The graph of a linear equation of the form $ax + by = c$ is a line which divides the plane into disjoint regions as stated below.

(1) The set of ordered pairs (x, y) such that $ax + by < c$.

(2) The set of ordered pairs (x, y) such that $ax + by > c$.

The regions (1) and (2) are called **half–planes**. The line $ax + by = c$ that divides the plane is called the **boundary** of both half planes. (See figure 9.1). If the boundary line is included in either plane then it is called **closed half plane**. Since a plane has infinite length and breadth, it cannot be completely shown by a figure. Only a segment of the plane has been shown in the figure.

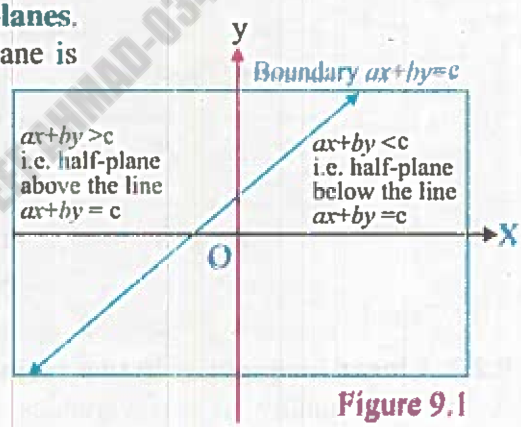


Figure 9.1

Note

A vertical line divides the plane into left and right half–planes while a non– vertical line divides the plane into upper and lower half–planes.

A **Solution** of a linear inequality in two variables x and y is an ordered pair of real numbers (a, b) such that the inequality is satisfied when we substitute $x = a$ and $y = b$.

For example, the ordered pair $(-1, 2)$ is a solution of the inequality $3x + y < 5$, since $3(-1) + 2 = -3 + 2 = -1 < 5$ which is true.

The graph of a linear inequality in two variables x and y is the set of all ordered pairs that satisfy the inequality.

Note

The graph of a single inequality, in more than two variables, is a half–plane.

9.2.2.2 Procedure for Graphing a Linear Inequality in Two variables

To graph a linear inequality, we follow the following procedure.

Step–1: Replace the inequality sign with an equal sign and draw the line. Make the line solid if the inequality involves $\leq$ or $\geq$, make the line dashed if the inequality involves $<$ or $>$.

Step–2: Choose any point that is not on the line as a test point. If the origin is not on the line, it is the most appropriate choice.

Step–3: Substitute the coordinates of the test point into the original inequality. If the test point satisfies the inequality, shade the half–plane that includes the test point, otherwise, shade the half–plane on the other sides of the line.

> **Did You Know**
>
> If a line intersects x–axis at $(a,0)$, then a is called x–intercept of the line.
> If a line intersects y–axis at $(0,b)$, then b is called y–intercept of the line.

Example 3: Graph the inequality $2x - 5y \geq 10$.

Solution: The associated equation of the inequality is

$$2x - 5y \geq 10 \qquad \text{(i)}$$

$$2x - 5y = 10 \qquad \text{(ii)}$$

Graph the line (ii) by finding x–and y–intercepts.

To find the x– intercept, let $y=0$.

To find y–intercept, let $x=0$.

We have $2x - 5(0) = 10$

$\Rightarrow \quad x = 5$

and $2(0) - 5y = 10$

$\Rightarrow \quad y = -2$

x	y
5	0
0	−2

Therefore, the boundary line passes through (5, 0) and (0, −2).

The line is solid because the inequality involves $\geq$.

We choose O(0,0) as a test point, because it is not on the line (ii)

Substituting $x=0, y=0$ into the original inequality,

$$2x - 5y \geq 10$$

we get $2(0) - 5(0) \geq 10$

$\Rightarrow \quad 0 \geq 10$

which is not true. Therefore the test point does not satisfy the inequality, and so the solution is not the half–plane that includes the origin.

Thus the solution is the half–plane not containing the origin (see figure 9.2).

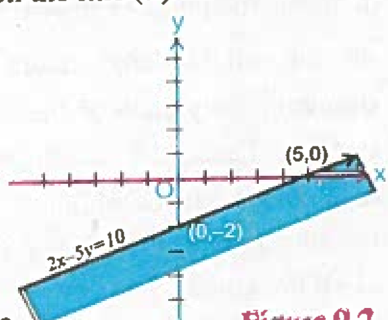


Figure 9.2

Example 4: Graph the inequality $y > x - 4$.

Solution: The associated equation of the inequality is

$y > x - 4$ (i)

$y = x - 4$ (ii)

x	y
4	0
0	-4

To find the x–intercept put y=0 in (ii)

$0 = x - 4 \Rightarrow x = 4$

Similarly to find the y–intercept put x=0 in (ii),

$y = 0 - 4$

$\Rightarrow y = -4$

Therefore the boundary line passes through (4, 0) and (0, –4). The line is dashed because the inequality involves >. We choose O (0, 0) as a test point, because it is not on the line (ii).

Substituting $x = 0$, y=0 into the original inequality

$y > x - 4$ we get $0 > 0 - 4$ $\Rightarrow 0 > -4$

which is true. Therefore the solution is the half–plane that includes the origin (see figure 9.3).

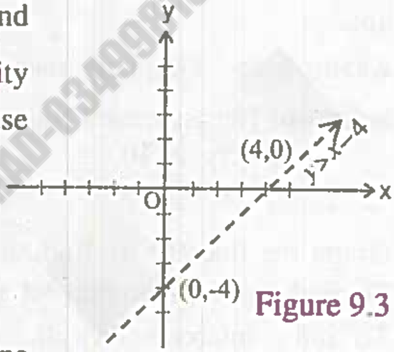


Figure 9.3

9.2.3. Region bounded by 2 or 3 simultaneous inequalities

(i.e. System of Linear Inequalities in Two Variables)

Two or more linear inequalities together form a **system of linear inequalities.** The graph of a system of linear inequalities in two variables x and y is the set of all ordered pairs (x, y) that satisfy simultaneously each of the linear inequalities in the system. Thus, the graph of a system of linear inequalities can be obtained by graphing each inequality individually and then taking intersection of all the graphs. The common region so obtained is called the **solution region** for the system of linear inequalities.

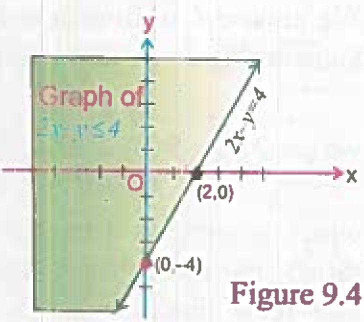


Figure 9.4

Example 5: Graph the system of linear inequalities.

$$\left.\begin{aligned} 2x - y &\le 4 \\ x + y &\ge 2 \end{aligned}\right\}$$

Solution: Following the procedure for graphing linear inequalities, the graph of the line $2x - y = 4$ is drawn by joining the points (2, 0) and (0, –4). The test point (0,0) satisfies the inequality, so the graph of the inequality $2x - y \le 4$ is the upper half–plane including the graph of the line $2x - y = 4$. The closed half–plane is partially shown as a shaded region in Figure 9.4.

The graph of the line $x + y = 2$ is drawn by joining the points (2, 0) and (0, 2). The test pint (0, 0) does not satisfy the original inequality, so the graph of the inequality $x + y \ge 2$ is the closed half–plane not on the origin side of the line $x+y=2$. The closed half–plane is partially shown by shading in the figure 9.5.

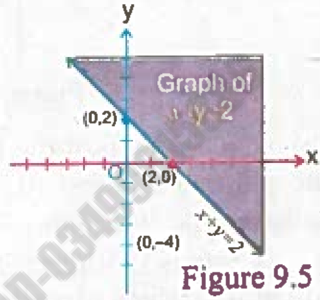


Figure 9.5

The solution region of the given system of linear inequalities is displayed in figure 9.6 by the shaded overlapping region of the graphs shown in figure 9.4 and figure 9.5. The point (2, 0) is the intersection point of the graph of the system of inequalities that can also be found by solving the equations $2x - y = 4$ and $x + y = 2$

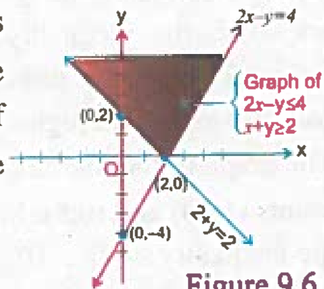


Figure 9.6

Example 6: Graph the solution region of the following system of linear inequalities in each case.

a) $\left.\begin{aligned} 2x - y &\le 4 \\ x + y &\ge 2 \\ -x + 2y &\le 4 \end{aligned}\right\}$ b) $\left.\begin{aligned} x - 2y &\le 6 \\ 2x + y &\ge 2 \\ x + 2y &\le 10 \end{aligned}\right\}$

Solution:(a) The graph of the inequalities $2x - y \le 4$ and $x + y \ge 2$ have already been plotted in figure 9.4 and figure 9.5 respectively and their solution region partially shown in figure 9.6 of example (5).

Following the procedure for graphing of linear inequalities, the graph of the inequality $-x + 2y \le 4$ is partially shown in figure 9.7.

The intersection of the three graphs is the required solution region which is the shaded triangular region ABC (including its sides) shown in figure 9.8.

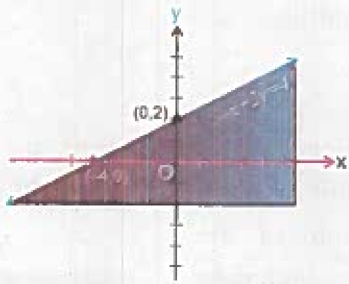


Figure 9.7

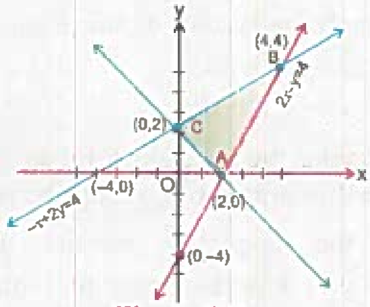


Figure 9.8

(b) The graph of the line $x-2y=6$ is drawn by joining the points (6,0) and (0,–3). Since the test point (0, 0) satisfies the inequality $x - 2y \le 6$, thus the graph of $x - 2y \le 6$ is the upper half–plane including the graph of the line which is partially shown by a shaded region in figure 9.9.

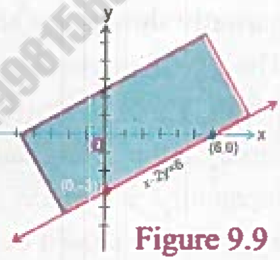


Figure 9.9

The graph of the line $2x + y = 2$ is drawn by joining the points (1, 0) and (0, 2). Since the test Point (0, 0) does not satisfy the inequality $2x + y \ge 2$, thus the graph of $2x+y\ge 2$ is the closed half–plane which is shown partially as shaded region in figure 9.10.

Figure 9.10

The graph of the line $x + 2y = 10$ is drawn by joining the points (10,0) and (0,5). Since the test point (0,0) satisfies the inequality $x+2y \le 10$, thus the graph of $x + 2y \le 10$ is the lower half–plane including the graph of the line which is partially shown by shading in figure 9.11.

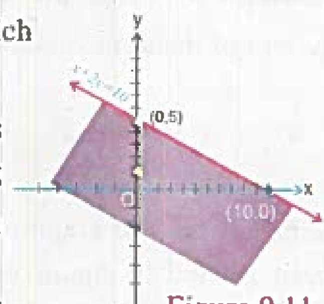


Figure 9.11

The required graph of the solution region of the system is the shaded overlapping triangular region ABC (including its sides) termed by the three graphs as shown in figure 9.12.

In example (6), we see that the solution region of either system is the shaded triangular region ABC as solution in figures 9.8 and 9.12 respectively where A, B and C are the points of the solution regions, obtained by the intersection of its boundary lines. Such points are termed as **corner points or vertices** of the solution region. Thus, a point of a solution region where two of its boundary lines intersect, is called a corner point or a vertex of the solution region. The corner points of the solution region can be obtained by solving the associated equations of linear inequalities in pairs.

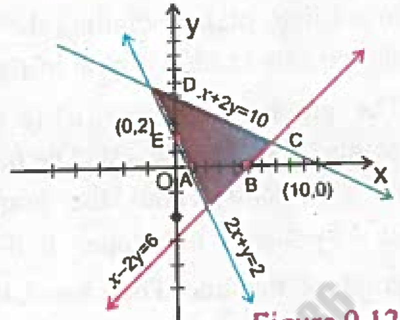


Figure 9.12

For example, in example 6 (a) the following three corner points are obtained by solving the associated equations of the inequalities in pairs.

Associated Equations of Inequalities		Corner Points
$2x - y = 4,$	$x + y = 2$	A (2, 0)
$2x - y = 4,$	$-x + 2y = 4$	B (4, 4)
$x + y = 2,$	$-x + 2y = 4$	C (0, 2)

The graph of a solution region of the system of linear inequalities may be either bounded or unbounded. The graph of the solution region is **bounded** if it can be enclosed within some circle of sufficiently large radius while the graph of the solution region is **unbounded**, if it cannot be enclosed in any circle how large its radius may be. In example (5), the solution region is unbounded while in example (6), the solution region of both systems (a) and (b) is bounded.

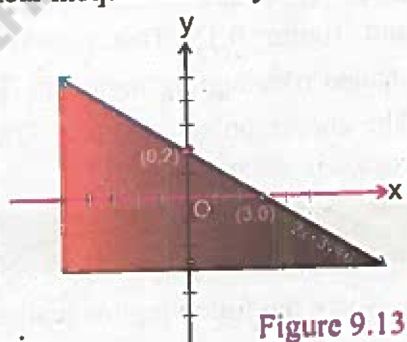


Figure 9.13

Example 7: Graph the solution region of the following system of linear inequalities and find their corner points. Also check whether the graph of the solution region is bounded or not.

$$\left.\begin{aligned} 2x + 3y &\le 6 \\ 2x - 3y &\le 6 \\ x &\ge 0 \end{aligned}\right\}$$

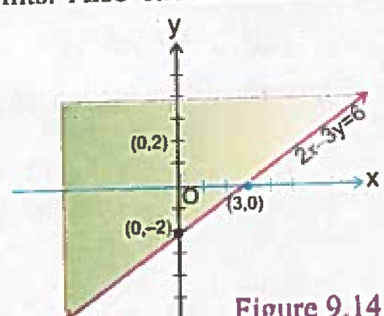


Figure 9.14

Solution: The associated equations of the linear inequalities

$$2x + 3y \le 6 \quad \text{and} \quad 2x - 3y \le 6$$

are $2x + 3y = 6$ (i) and $2x - 3y = 6$ (ii)

The graph of line (i) is drawn by joining the points (3, 0) and (0, 2). The test point (0, 0) satisfies the inequality. Thus the graph of the inequality $2x + 3y \le 6$ is the lower half plane including the graph of the line. The closed half plane is partially shown as a shaded region in figure 9.13.

The graph of the line (ii) is drawn by joining the points (3, 0) and (0, –2). The test point (0, 0) satisfies the inequality. Thus the graph of the inequality $2x - 3y \le 6$ is the upper half–plane including the graph of the line. The closed half plane is partially shown by a shaded region in figure 9.14.

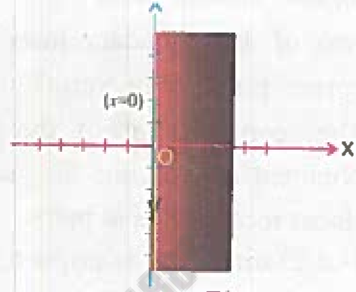


Figure 9.15

The graph of $x \ge 0$ is the right half–plane including the graph of the line $x = 0$ (the y–axis) of the linear inequality $x \ge 0$. The graph of $x \ge 0$ is partially shown in figure 9.15. The solution region of the given system of linear inequalities is the intersection of the graph partially shown in figure 9.13, figure 9.14 and figure 9.15. This region is displayed as the shaded overlapping region in (Figure 9.16).

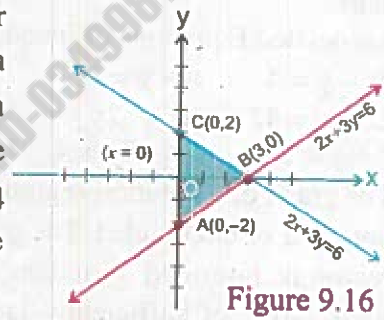


Figure 9.16

The corner points are A(0, –2), B(3, 0) and C(0,2). The graph of the solution region is clearly bounded.

EXERCISE 9.1

1. Solve the following inequalities and graph the solution set in each case

(i) $x + 3 < 7$ (ii) $-3x - 2 \le 4$

(iii) $x + y \le 2$ (iv) $2x - 3y > 6$

2. Graph the following systems of linear inequalities.

(i) $\left.\begin{aligned} 2x - 3y &\le 12 \\ 3x + 2y &\le 6 \end{aligned}\right\}$ (ii) $\begin{aligned} x - y &\le 1 \\ x + y &\ge 4 \end{aligned}$

(iii) $\left.\begin{aligned} 2x + y &\ge 4 \\ x + y &\ge 3 \\ x &\ge 0 \end{aligned}\right\}$ (iv) $\left.\begin{aligned} 2x + y &\le 8 \\ x + y &\le 6 \\ y &\ge 0 \end{aligned}\right\}$

3. Graph the solution region of the following system of linear inequalities and find the corner points in each case. Also tell whether the graph is bounded or unbounded.

(i) $\left.\begin{aligned} 2x + y &\le 6 \\ x + 2y &\le 6 \\ x &\ge 0 \end{aligned}\right\}$ (ii) $\left.\begin{aligned} 2x + 3y &\ge 6 \\ x + y &\ge 4 \\ y &\ge 0 \end{aligned}\right\}$

4. Graph the solution region of the following system of linear inequalities and find the corner points in each case. Also tell whether the graph is bounded or unbounded.

(i) $\left.\begin{aligned} 2x + 3y &\le 12 \\ 3x + y &\le 12 \\ x + y &\ge 2 \end{aligned}\right\}$ (ii) $\left.\begin{aligned} 2x + y &\ge 3 \\ x + y &\le 5 \\ x - y &\ge 2 \end{aligned}\right\}$

9.3 Feasible region

9.3.1 Define linear programming problem, objective function, problem constraints and decision variables

As mentioned earlier, linear programming consists of methods for finding the maximum or minimum value of a linear function in two variables of the form $f(x, y) = ax + by$; $a, b \in \mathbb{R}$,

where the variables x and y are subject to the set of conditions or constraints given in the form of linear inequalities. In order to maximize or minimize the linear function $f(x, y) = ax + by$, called the **objective function,** we need to find points (x, y) that make the function largest (or smallest) possible. Such points occur at the corner of the feasible region as the following theorem asserts.

"The maximum (or minimum) value of the objective function is achieved at one of the corner of the feasible region."

Many practical problems arising in the field of business, economics, the sciences and engineering involve systems of linear inequalities. In such problems the choice of values of the variables is not entirely free but subject to some restrictions or conditions given in the form of linear inequalities. The linear inequalities involved in the problem are called **problem constraints.** The variables used in the system of linear inequalities relating to the problem are non–negative and called **non–negative constraints** or **decision variables**.

The graph of the solution region of the system of linear inequalities

$x-2y\leq 6$
$2x+y\geq 2$
$x+2y\leq 10$

is given in (Figure 9.17). We observe that the solution region of the system of linear inequalities is not always within the first quadrant. However, the solution region can be restricted to the first quadrant if the non–negative constraints $x\geq 0$, $y\geq 0$ are included in the system of linear inequalities. In example 6 (b), if $x\geq 0$ and $y\geq 0$ are included within the system of linear inequalities, then the solution region can be restricted to the first quadrant. It is the polygonal region ABCDE (including its sides) as shown in Figure 9.18

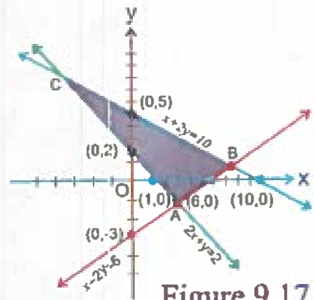


Figure 9.17

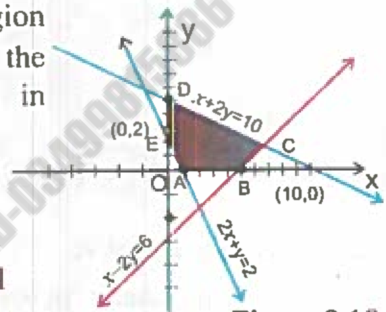


Figure 9.18

9.3.2 A region (which is restricted to the first quadrant) is referred to as a **feasible region**. Each point of the feasible region is called **feasible solution** of the system of linear inequalities (or for the set of given constraints). In other words any ordered pair (x, y) that satisfies all the constraints is called a **feasible solution** of the system of linear inequalities and the set of all feasible solutions is called a **feasible solution set.**

Example 8: Graph the feasible region of the following system of linear inequalities.

$$\left.\begin{aligned} 3x + 5y &\leq 15 \\ -x + 3y &\leq 3 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}\right\}$$

Solution: The associated equations for the inequalities

$3x+5y\leq 15$ and $-x+3y\leq 3$

are $3x+5y=15$ (i) and $-x+3y=3$ (ii)

The graph of line (i) is drawn by joining the points (5, 0) and (0, 3) by a solid line.

Similarly, the graph of line (ii) is drawn by joining the points (–3, 0) and (0,1) by a solid line.

Since the test point (0, 0) satisfies both the inequalities $3x+5y\leq 15$ and $-x+3y\leq 3$, so both the closed half–planes are on the origin sides of line (i) and (ii).

The intersection of these closed half– planes is the shaded overlapping region as shown in Figure 9.19

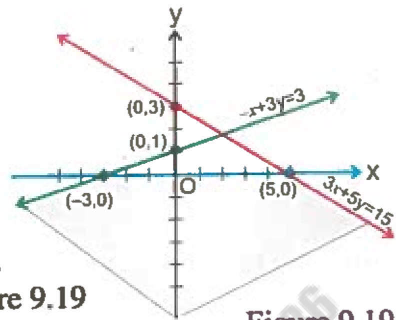


Figure 9.19

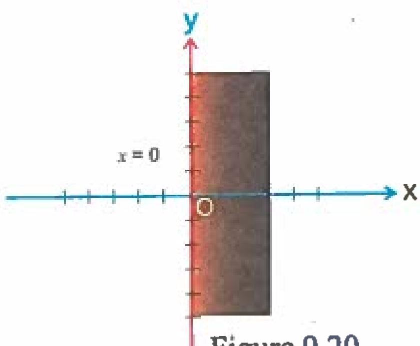


Figure 9.20

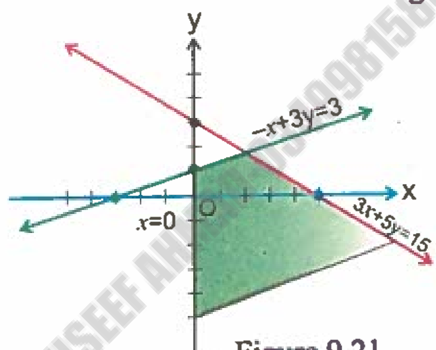


Figure 9.21

The graph of $x\geq 0$ is partially shown in Figure 9.20. The intersection of graphs shown in Figure 9.19 and Figure 9.20 is partially displayed as a shaded region in Figure 9.21.

The graph of $y\geq 0$ is partially plotted in Figure 9.22.

The intersection of graphs shown in Figure 9.19 and Figure 9.22 is partially displayed as shaded region in Figure 9.23.

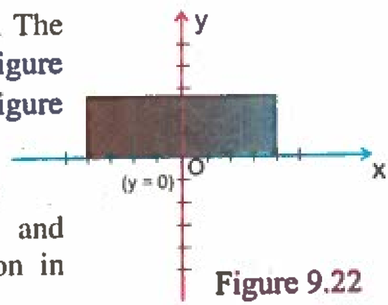


Figure 9.22

The graph of the given system of linear inequalities is the intersection of the graphs shown in Figure 9.21 and Figure 9.23 which is indicated as shaded region in Figure 9.24. This shaded region is the required feasible region of the given system of linear inequalities.

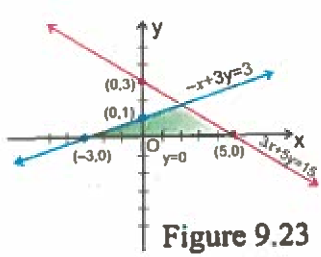


Figure 9.23

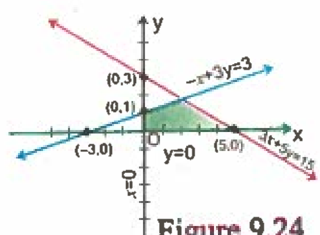


Figure 9.24

Example 9: Graph the feasible region subject to the following constraints.

(a) $\left.\begin{aligned} 3x - 4y &\le 12 \\ 3x + 2y &\ge 6 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}\right\}$ (b) $\left.\begin{aligned} 3x - 4y &\le 12 \\ 3x + 2y &\ge 6 \\ x + 2y &\le 10 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}\right\}$

Did You Know ?

The feasible solution region in example 9(a) is unbounded while the feasible solution region in example 9(b) is bounded.

Solution: (a) The associated equations for the inequalities $3x - 4y \le 12$ and $3x + 2y \ge 6$

are $3x - 4y = 12$ (i) and $3x + 2y = 6$ (ii)

The graph of line (i) is drawn by joining the points (4, 0) and (0, –3) by a solid line. Since the test point (0, 0) satisfies the inequality $3x - 4y \le 12$, so the graph of $3x - 4y \le 12$ is the closed half–plane on the origin side of line $3x - 4y = 12$. The graph of system

$$3x - 4y \le 12$$
$$x \ge 0$$
$$y \ge 0$$

is partially shown as shaded region in Figure 9.25.

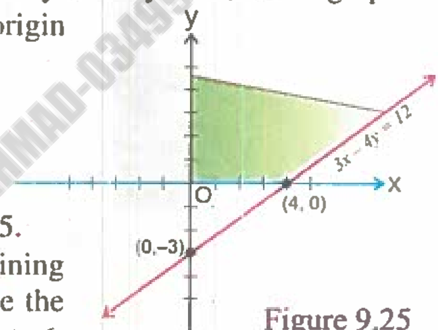


Figure 9.25

Similarly, the graph of line (ii) is drawn by joining the points (2, 0) and (0, 3) by a solid line. Since the test point does not satisfy the inequality $3x + 2y \ge 6$, so the graph of $3x + 2y \ge 6$ is the closed half–plane not on the origin side of the line $3x + 2y = 6$. The graph of system

$$3x + 2y \ge 6$$
$$x \ge 0$$

$y \ge 0$ is partially drawn as shaded region in Figure 9.26.

The graph of the system

$$3x - 4y \le 12$$
$$3x + 2y \ge 6$$
$$x \ge 0$$
$$y \ge 0$$

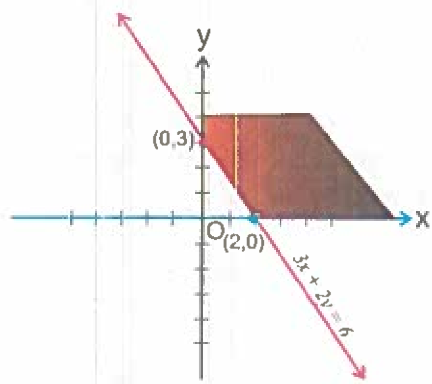


Figure 9.26

is the intersection of the graphs shown in (Figure 9.25) and figure 9.26 and it is partially displayed in (Figure 9.27) as shaded region. This shaded region in the graph of the feasible region subject to the given constraints.

Corner Points: (2, 0), (4, 0), (0, 3)

(b) The graph of the system

$3x - 4y \leq 12$

$3x + 2y \geq 6$

$x \geq 0$

$y \geq 0$ is partially shown in Figure 9.27

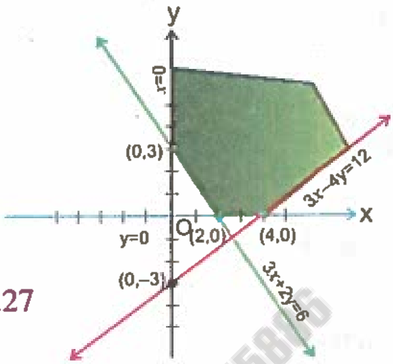


Figure 9.27

The graph of the system

$x + 2y \leq 10$

$x \geq 0$

$y \geq 0$ is shown by shaded region in figure 9.28

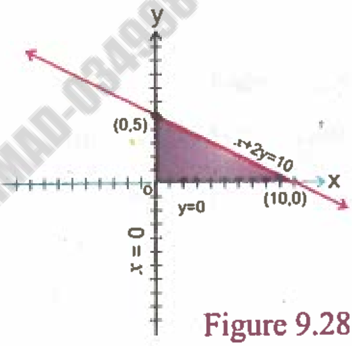


Figure 9.28

The graph of the system

$3x - 4y \leq 12$

$3x + 2y \leq 6$

$x + 2y \leq 10$

$x \geq 0$

$y \geq 0$ is the intersection of the graphs shown in figure 9.27 and figure 9.28 and it is indicated in figure 9.29 as shaded region.

Corner Points:

(2, 0), (4, 0), $\left(\frac{4}{5}, \frac{9}{5}\right)$, (0, 5) (0, 3)

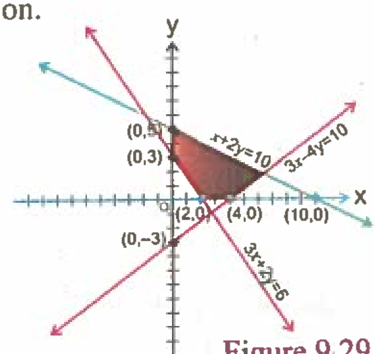


Figure 9.29

9.4. Optimal solution

9.4.1 There are infinitely many feasible solutions in the feasible region. The feasible solution which maximizes or minimizes the objective function is called the **Optimal Solution.**

The procedure for finding the optimal solution (maximum or minimum value) of the objective function $f(x, y) = ax + by$, subject to a set of linear constraints (inequalities) in variables x and y is as following:

9.4.2 Procedure for determining optimal solution

Step–1: Determine the feasible region by graphing the linear inequalities that form the constraints.

Step–2: Find the corner points of feasible region by solving two equations at a time of the boundary lines of the feasible region.

Step–3: Compute the value of the objective function $f(x,y) = ax+by$ at each of the corner points.

Step–4: To find the optimal solution, select the largest value computed in step-3 if $f(x,y) = ax+by$ has to be maximized, and select the smallest value if $f(x, y) = ax + by$ has to be minimized.

Example 10: Find the maximum and minimum values of the function $f(x, y) = 2x + 3y$ subject to the constraints

$$\begin{aligned} 3x - y &\geq -1 \\ x + y &\leq 5 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Solution: The graph of the inequality $3x - y \geq -1$ is the closed half–plane on the origin side of the line $3x - y = -1$ and the graph of the inequality $x + y \leq 5$ is the closed half–plane also on the origin side of the line $x + y = 5$.
The graph of the system

$$\begin{aligned} 3x - y &\geq -1 \\ x + y &\leq 5 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

is shown as a shaded region in Figure 9.30. This shaded region is the feasible region. We see that the feasible region is bounded and its corner points are O(0,0), A (5, 0), B(1, 4) and C (0,1). Evaluating the given function $f(x, y)$ at the corner points, we get

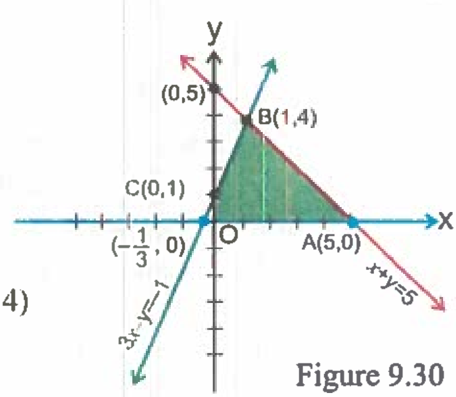


Figure 9.30

$$f(0, 0) = 2(0) + 3(0) = 0$$
$$f(5, 0) = 2(5) + 3(0) = 10$$
$$f(1, 4) = 2(1) + 3(4) = 14$$
$$f(0, 1) = 2(0) + 3(1) = 3$$

Thus the minimum value of $f(x, y)$ is 0 at the corner point O(0, 0) and the maximum value of $f(x, y)$ is 14 at the corner point (1, 4).

Example 11: Find the maximum and minimum value of the function $f(x,y) = 4x + 2y$ subject to the constraints

$$x + 2y \leq 8$$
$$x + y \leq 5$$
$$2x + y \leq 8$$
$$x \geq 0$$
$$y \geq 0$$

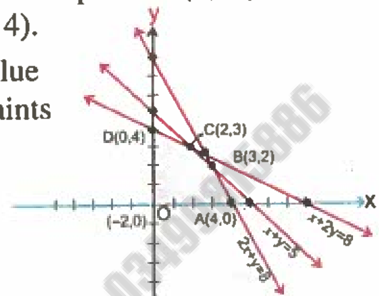


Figure 9.31

Solution: The solution region of the system

$$x + 2y \leq 8$$
$$x + y \leq 5$$
$$2x + y \leq 8$$
$$x \geq 0$$
$$y \geq 0$$

> **Note**
>
> In example 11, the function $f(x, y)$ has maximum value at two corner points (4,0) and (3,2). It follows that $f(x, y)$ has maximum value at all the points of the line segment between the points (3,2) and (2,3).

is the shaded region OABCD shown in figure 9.31. We see that the feasible region is bounded and its corner points are O(0, 0), A (4, 0), B (3, 2), C(2,3) and D (0, 4). We compute the values of the function $f(x, y)$ at the corner points to find its maximum and minimum values. The value of $f(x, y)$ at the corner points are given in the following table.

Corner Points	$f(x, y) = 4x + 2y$
(0, 0)	$f(0, 0) = 4(0) + 2(0) = 0$
(4, 0)	$f(4, 0) = 4(4) + 2(0) = 16$
(3, 2)	$f(3, 2) = 4(3) + 2(2) = 16$
(2, 3)	$f(2, 3) = 4(2) + 2(3) = 14$
(0, 4)	$f(0, 4) = 5(0) + 2(4) = 8$

From the above table, we see that the minimum value of the function $f(x,y)$ is 0 at the corner point (0, 0) and the maximum value of $f(x, y)$ is 16 at the corner points (4, 0) and (3,2).

9.4.3. Real life LP Problems

To solve a linear programming problem, first formulate a mathematical model of the problem and then use the procedure given in **section** 9.4 to solve it.

Mathematical formulation of a linear programming problem

The mathematical formulation of a linear programming problem involves the following basic steps:

Step 1 Identify the decision variable and assign symbol x and y to them. These decision variables are those quantities whose values we wish to determine.

Step 2 Identify the set of constraints and express them as linear equations / inequations in terms of the decision variables. These constraints are the given conditions.

Step 3 Identify the objective function and express it as a linear function of decision variables. It might take the form of maximizing profit or production or minimizing cost.

Step 4 Add the non-negativity restrictions on the decision variables, as in the physical problems, negative values of decision variables have no valid interpretation.

Example 12: A furniture dealer deals in only two items, viz., tables and chairs. He has Rs. 10,000 to invest and a space to store at most 60 pieces. A table costs him Rs. 500 and chair Rs. 200. He can sell a table at a profit of Rs. 50 and a chair at a profit of Rs. 15. Assume that he can sell all the items that he buys. Formulate this problem as on LPP, so that he can maximize the profit.

Solution: Let x and y denote the number of tables and chairs respectively (x and y are decision variables).

The cost of x tables = Rs. $500x$, The cost of y tables = Rs. $200y$

Therefore, the total cost of x tables and y chairs = Rs. $(500x + 200\ \ )$, which cannot be more than 10000. Thus $500x + 200y \le 10000$ (Constraint)

Also, $x + y \le 60$ (constraint) as the dealer has the space to store at the most 60 items. It is obvious that $x \ge 0$, $y \ge 0$ (non-negative restrictions) as the number of tables and chairs cannot be negative.

Profit on x tables = $50x$, Profit on y chairs = $15y$

Hence, the profit on x tables and y chairs = Rs. $50x + 15y$ (objective function).

Obviously, the dealer wishes to maximize the profit $Z = 50x + 15y$

Thus, the mathematical formulation of the LPP is

Maximize $Z = 50x + 15y$ subject to the constraints

$$5x + 2y \le 100$$
$$x + y \le 60$$
$$x \ge 0,\ \ y \ge 0$$

Example 13: A factory produces two types of food containers A and B by using two machines M_1 and M_2. To produce container A, M_1 works 2 minutes and M_2, 4 minutes. Similarly, to produce container B, M_1, works 8 minutes and M_2, 4 minutes. The profit for container A is Rs. 29 and for B is Rs. 45. How many container of each type should be produced so that a maximum profit can be achieved?

Solution: Let x = the number of container A per minute and

y = the number of container B per minute.

If per hour production of M_1 and M_2 is x container A and y container B, then the profit per hour is given by the profit function $P(x, y) = 29x + 45y$.

The constraints are

$2x + 8y \leq 60$ (Resulting from machine M_1)

$4x + 4y \leq 60$ (Resulting from machine M_2)

$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \end{array}\right\}$ (since container cannot be negative)

The above system of linear inequalities/ constraints can be written in the following simplified form

$$x + 4y \leq 30$$
$$x + y \leq 15$$
$$x \geq 0$$
$$y \geq 0$$

We maximize the profit function P under the given constraints.

As before, graphing the linear inequalities, we obtain the feasible region OABC which is shaded in Figure 9.32. Solving the equations $x + 4y = 30$ and $x + y = 15$, we get $x = 10$, $y = 5$, that is, their point of intersection is (10, 5). Thus, the corner points of the feasible region are O(0,0), A(15,0), B(10,5) and C $\left(0, \frac{30}{4}\right)$.

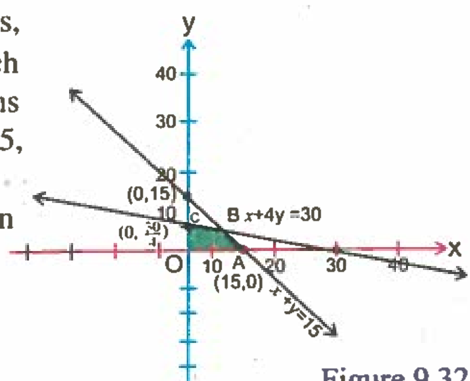


Figure 9.32

We find the value of the function P at the corner points.

Corner Points	P (x, y) = 29x + 45y
(0, 0)	P (0, 0) = 29 (0) + 45 (0) = 0
(15, 0)	P (15, 0) = 29 (15) + 45 (0) = 435
(10, 5)	P (10, 5) = 29 (10) + 45 (5) = 515
$(0, \frac{30}{4})$	$P(0, \frac{30}{4}) = 29(0) + 45(\frac{30}{4}) = 337.50$

From the above table, we see that the maximum profit is Rs. 515 per hour at the corner point B (10, 5). Thus, the optimal production plan that maximizes the profit is achieved by producing 10 containers of A and 5 containers of B.

Example 14: A farmer possesses 80 acres of land and wish to grow two types of crops A and B. Cultivation of crop A requires 3 hours per acres and cultivation of crop B requires 2 hours per acres. Working hours cannot exceed 180. If he gets a profit of Rs. 50 per acres for crop A and Rs.40 per acre for crop B, then how many acres of each crop should be cultivated to maximize his profit.

Solution: Let x = Acres required for cultivation of crop A

and y = Acres required for cultivation of crops B.

If P(x, y) is the profit function, then

$$P(x, y) = 50x + 40y$$

The constraints are

$x + y \leq 80$ (Restriction of land)

$3x + 2y \leq 180$ (Restriction due to time)

$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \end{array}\right\}$ (Non–negative constraints, since acres cannot be negative)

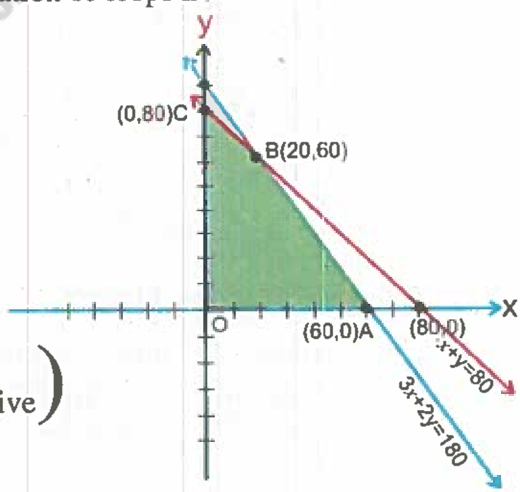


Figure 9.33

Graphing the inequalities, we obtain the feasible region OABC which is displayed by shading in figure 9.33. Solving $x + y = 80$ and $3x + 2y = 180$, we get x=20 and y= 60, that is their point of intersection is (20, 60). Thus the corner points of the feasible region are O(0, 0), A (60, 0), B (20, 60) and C (0, 80). We find the values of the function P at the corner points.

Corner Points	$P(x, y) = 50x + 40y$
(0, 0)	$P(0, 0) = 50(0) + 40(0) = 0$
(60, 0)	$P(60, 0) = 50(60) + 40(0) = 3000$
(20, 60)	$P(20, 60) = 50(20) + 40(60) = 3400$
(0, 80)	$P(0, 80) = 50(0) + 40(80) = 3200$

From the above table, we see that the maximum profit is Rs. 3400 at the corner point (20, 60). Thus, the farmer will get the maximum profit if he cultivates 20 acres of crop A and 60 acres of crop B.

EXERCISE 9.2

1. Graph the feasible region of the following system of linear inequalities and also find the corner points.

(i) $\left.\begin{aligned} 2 + y &\le 6 \\ 4x + y &\le 8 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}\right\}$ (ii) $\left.\begin{aligned} 3x - y &\ge -4 \\ x + y &\le 5 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}\right\}$

(iii) $\left.\begin{aligned} 2x + y &\ge 6 \\ 2x + 3y &\le 12 \\ -x + y &\le 2 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}\right\}$ (iv) $\left.\begin{aligned} x + y &\ge 3 \\ 2x + 3y &\le 12 \\ x - y &\le 2 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}\right\}$

2. (i) Maximize $f(x, y) = 2x + y$ subject to the constraints

$$x + y \le 6$$
$$x + y \ge 1$$
$$x, y \ge 0$$

(ii) Maximize $f(x, y) = 3x + 5y$ subject to the constraints

$$2x + 3y \le 12$$
$$3x + 2y \le 12$$
$$x + y \ge 2$$
$$x \ge 0$$
$$y \ge 0$$

3. (i) Find the maximum and minimum values of the function $f(x,y)=5x+2y$ subject to the constraints

$$\begin{aligned} 2x + y &\ge 2 \\ x + 2y &\le 10 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

(ii) Find the maximum and minimum values of the function $f(x,y)=7x+21y$ subject to the constraints

$$\begin{aligned} 2x + y &\ge 2 \\ 2x + 3y &\le 6 \\ x + 2y &\le 8 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

4. A company manufactures two models of bicycles, model A and model B by using two machines M_1 and M_2. Machine M_1 has at most 120 hours available and machine M_2 has a maximum of 144 hours available. Manufacturing a model A bicycle requires 5 hours in machine M_1 and 4 hours in machine M_2 and manufacturing of a model B bicycle requires 4 hours in machine M_1 and 8 hours in machine M_2. If the company gets profit of Rs. 40 per model A bicycle and profit of Rs. 50 per model B bicycle, how many of each model should be manufactured for maximum profit?

5. A machine can produce product A by using 2 units of chemical and 1 unit of a compound or can produce product B by using 1 unit of chemical and 2 units of the compound. Only 800 units of chemical and 1000 units of the compound are available. The profit per unit of A and B are Rs. 30 and Rs. 20 respectively. Determine how many units of each product should be produced to achieve the maximum profit.

6. A company manufactures and sells two models of lamps, L_1, L_2, Use the following table to determine how many of each type of lamps should be produced to achieve a maximum profit?

	Model L_1	Model L_2	Maximum Time Available
Manufacturing time per lamp	2 hours	1 hour	40 hours
Finishing time per lamp	1 hour	1 hour	32 hours
Profit per lamp.	Rs. 70	Rs. 50	—

REVIEW EXERCISE 9

1. Choose the correct option

(i) The solution of the system of inequalities $x \geq 0$, $x - 5 \leq 0$ and $x \geq y$ is a polygonal region with the vertices as

(a) (0,0), (5.0), (5,5) (b) (0,0), (0,5), (5,5)

(c) (5,5), (0,5), (5,0) (d) (0,0), (0,5), (5,0)

(ii) Find the profit function p if it yields the value 11 and 7 at (3,7) and (1,3) respectively

(a) $P = -8x + 5y$ (b) $p = 8x - 5y$

(c) $p = 8x + 5y$ (d) $p = -(8x + 5y)$

(iii) The vertices of closed convex polygon representing the feasible region of the objective function are (6, 2), (4, 6), (5, 4) and (3, 6). Find the maximum value of the function $f = 7x + 11y$

(a) 64 (b) 79 (c) 94 (d) 87

(iv) Which of the following is a point in the feasible region determined by the linear inequations $2x + 3y \leq 6$ and $3x - 2y \leq 16$?

(a) (4, −3) (b) (−2, 4) (c) (3, −2) (d) (3, −4)

(v) The maximum value of the function $f = 5x + 3y$ subjected to the constraints $x \geq 3$ and $y \geq 3$ is ____________

(a) 15 (b) 9 (c) 24 (d) does not exist

(vi) Maximize $5x + 7y$, subject to the constraints $2x + 3y \leq 12$, $x + y \leq 5$, $x \geq 0$ and $y \geq 0$

(a) 29 (b) 30 (c) 28 (d) 31

2. Maximize $Z = 4x + 3y$ subject to the constraints

$$\left.\begin{aligned} 3x + 4y &\leq 24 \\ 8x + 6y &\leq 48 \\ x &\leq 5 \\ y &\leq 6 \\ x, y &\geq 0 \end{aligned}\right\}$$

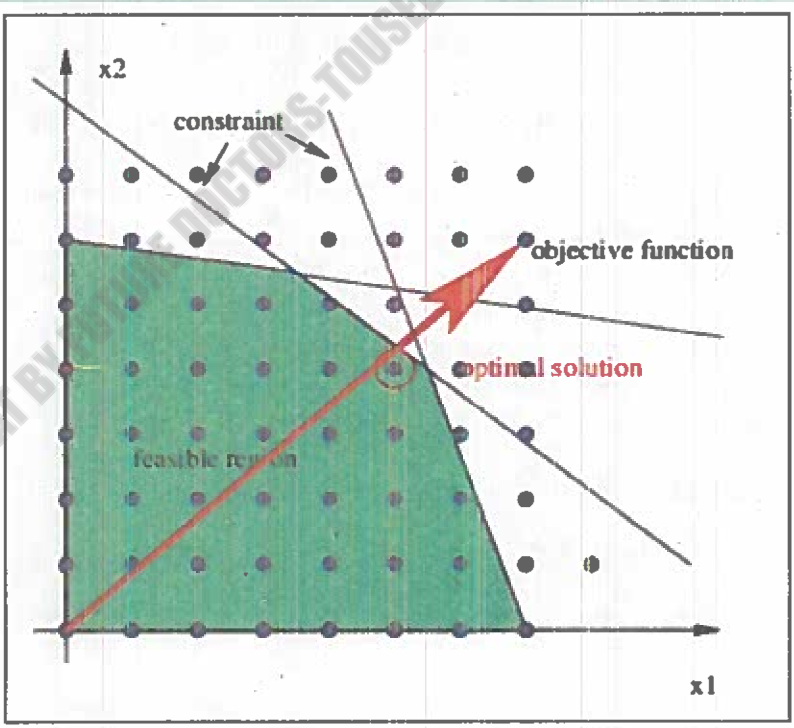
3. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains atleast 10 units of vitamin A, 12 units of vitamins B and 8 units of vitamin C. The vitamin content of one kg, food is given below:

Food	Vitamin A	Vitamin B	Vitamin C
X	1	2	3
Y	2	2	1

One kg of food X costs Rs.16 and one kg of food Y costs Rs.20. Find the least cost of the mixture which will produce the required diet.

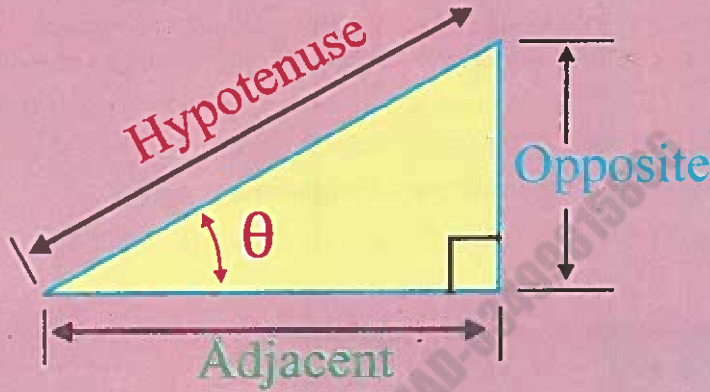
4. Find the maximum and minimum values of the function $Z = 5x + 10y$ subject to the constraints

$$x + 2y > 120$$
$$x + y > 60$$
$$x - 2y > 0$$
$$x, y > 0$$



UNIT 10

TRIGONOMETRIC IDENTITIES OF SUM AND DIFFERENCE OF ANGLES



STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Use distance formula to establish fundamental law of trigonometry
  - $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$, and deduce that
  - $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$,
  - $\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$,
  - $\tan(\alpha\pm\beta)=\dfrac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$
- Define allied angles
- Use fundamental law and its deductions to derive trigonometric ratios of allied angles
- Express $a\sin\theta+b\cos\theta$ in the form $r\sin(\theta+\varphi)$ where $a=r\cos\varphi$ and $b=r\sin\varphi$
- Derive double angle, half angle and triple angle identities from fundamental law and its deductions.
- Express the product (of sines and cosines) as sums or differences (of sines and cosines)
- Express the sums or differences (of sines and cosines) as products (of sines and cosines)

10.1 Introduction

In the previous class some basic trigonometric identities have been proved and applied to show different trigonometric relations. This unit is a continuation of derivations of different trigonometric identities. These identities play an important role in calculus, the physical and life sciences and economics, where these identities are used to simplify complicated expressions.

We shall first establish the fundamental law of trigonometry so as to be able to deduce other trigonometric identities.

10.1.1 Fundamental law of trigonometry

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$.......(1)

Consider the given unit circle with center at O.

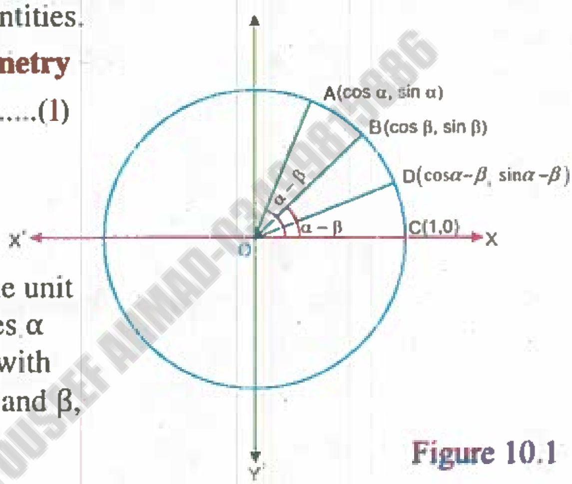


Figure 10.1

To establish the identity (1), we use the unit circle shown in **Figure 10.1**. The angles α and β are drawn in standard position, with $\overline{OA}$ and $\overline{OB}$ as the terminal sides of α and β, respectively.

The coordinates of A are (cosα, sinα),

The coordinates of B are(cosβ, sinβ).

The angle (α−β) is formed by the terminal sides of the angles α and β. An angle equal in measure to angle (α−β) is placed in standard position in the same figure $(\angle COD)$.

From geometry, if two central angles of a circle have the same measure, then the respective chords are also equal in measure. Thus the chords $\overline{AB}$ and $\overline{CD}$ are equal in length. Using the distance formula, we can calculate the lengths of the chords $\overline{AB}$ and $\overline{CD}$.

The length of a line segment with end points (x_1, y_1) and (x_2, y_2) is given by the following distance formula

$d=|P_1P_2|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

We apply this formula to the chords $\overline{AB}$ and $\overline{CD}$.

As $|\overline{AB}|=|\overline{CD}|$, so by distance formula,

$$\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2}=\sqrt{[\cos(\alpha-\beta)-1]^2+[\sin(\alpha-\beta)]^2}$$

Squaring each side of the equation and simplifying, we obtain

$$(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=\left[\cos(\alpha-\beta)-1\right]^2+\left[\sin(\alpha-\beta)\right]^2$$

$$\Rightarrow \cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta$$

$$=\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)$$

$$\Rightarrow \cos^2\alpha+\sin^2\alpha+\cos^2\beta+\sin^2\beta-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta$$

$$=\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta)+1-2\cos(\alpha-\beta)$$

Simplifying by using $\sin^2\theta+\cos^2\theta=1$, we have

$$2-2\sin\alpha\sin\beta-2\cos\alpha\cos\beta=2-2\cos(\alpha-\beta).$$

Solving for $\cos(\alpha-\beta)$, it gives us

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

We refer to (1) as fundamental law of trigonometry.

10.1.2. Deductions from the fundamental law of trigonometry

The following can be deduced from the fundamental law of trigonometry which are useful and play a significant role in proving the other trigonometry identities.

(i) $\cos(-\beta)=\cos\beta$

By Fundamental Law of Trigonometry,

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Letting $\alpha=0$, we get

$$\cos(0-\beta)=\cos 0\cos\beta+\sin 0\sin\beta$$

$$\cos(-\beta)=1\cdot\cos\beta+0\cdot\sin\beta$$

$$\cos(-\beta)=\cos\beta$$

(ii) $\cos(\frac{\pi}{2}-\beta)=\sin\beta$

By Fundamental Law of Trigonometry,

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Letting $\alpha=\frac{\pi}{2}$, we get

$$\cos(\frac{\pi}{2}-\beta)=\cos\frac{\pi}{2}\cos\beta+\sin\frac{\pi}{2}\sin\beta$$

$$\Rightarrow \cos(\frac{\pi}{2}-\beta)=0\cdot\cos\beta+1\cdot\sin\beta \qquad \left(\because \cos\frac{\pi}{2}=0, \sin\frac{\pi}{2}=1\right)$$

$$\therefore \cos(\frac{\pi}{2}-\beta)=\sin\beta$$

(iii) $\sin(\frac{\pi}{2}+\alpha)=\cos\alpha$

By identity (2), $\cos(\frac{\pi}{2}-\beta)=\sin\beta$

Letting $\beta=\frac{\pi}{2}+\alpha$, we get

$$\cos\left(\frac{\pi}{2}-\left(\frac{\pi}{2}+\alpha\right)\right)=\sin\left(\frac{\pi}{2}+\alpha\right) \Rightarrow \cos(-\alpha)=\sin\left(\frac{\pi}{2}+\alpha\right)$$

$$\Rightarrow \cos\alpha=\sin\left(\frac{\pi}{2}+\alpha\right) \; (\because \cos(-\alpha)=\cos\alpha)$$

$$\therefore \sin(\frac{\pi}{2}+\alpha)=\cos\alpha$$

(iv) $\cos(\frac{\pi}{2}+\alpha)=-\sin\alpha$

By Fundamental Law of Trigonometry,
$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Letting $\beta=-\frac{\pi}{2}$, we get

$$\cos(\alpha-(-\frac{\pi}{2}))=\cos\alpha\cos(-\frac{\pi}{2})+\sin\alpha\sin(-\frac{\pi}{2})$$

$$\Rightarrow \cos(\alpha+\frac{\pi}{2})=\cos\alpha\cdot 0+\sin\alpha(-1)$$
$$\left(\because \cos(-\frac{\pi}{2})=\cos\frac{\pi}{2}=0, \sin(-\frac{\pi}{2})=-\sin\frac{\pi}{2}=-1\right)$$

$$\therefore \cos(\frac{\pi}{2}+\alpha)=-\sin\alpha$$

(v) $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

By Fundamental law of trigonometry,
$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Replacing β by $-\beta$, we get

$\cos(\alpha-(-\beta)) = \cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta)$

$\therefore \cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta$ $\quad (\because \cos(-\beta)=\cos\beta, \sin(-\beta)=-\sin\beta)$

(vi) $\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$

By identity (5), $\cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta$

Replacing α by $\frac{\pi}{2}+\alpha$, we get

$$\cos\left(\left(\frac{\pi}{2}+\alpha\right)+\beta\right) = \cos\left(\frac{\pi}{2}+\alpha\right)\cos\beta-\sin\left(\frac{\pi}{2}+\alpha\right)\sin\beta$$

$$\Rightarrow \cos\left(\frac{\pi}{2}+(\alpha+\beta)\right) = \cos\left(\frac{\pi}{2}+\alpha\right)\cos\beta-\sin\left(\frac{\pi}{2}+\alpha\right)\sin\beta$$

By using identities (3) and (4), we get

$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$

(vii) $\sin(\alpha-\beta) = \sin\alpha\cos\beta-\cos\alpha\sin\beta$

By identity (6), $\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$

Replacing β by $-\beta$, we get

$\sin(\alpha+(-\beta)) = \sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$

$\Rightarrow \sin(\alpha-\beta) = \sin\alpha(\cos\beta)+\cos\alpha(-\sin\beta)$

$(\because \cos(-\beta)=\cos\beta, \sin(-\beta)=-\sin\beta)$

$\therefore \sin(\alpha-\beta) = \sin\alpha\cos\beta-\cos\alpha\sin\beta$

(viii) $\tan(-\theta) = -\tan\theta$

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

(ix) $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

Dividing numerator and denominator of R.H.S by $\cos\alpha\cos\beta$,

$$\tan(\alpha+\beta)=\frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}=\frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$=\frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{\sin\beta}{\cos\beta}}$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

(x) $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

By identity (ix), $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

Replacing β by $-\beta$, we get

$$\tan(\alpha+(-\beta))=\frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)}$$

$$\Rightarrow\ \tan(\alpha-\beta)=\frac{\tan\alpha+(-\tan\beta)}{1-\tan\alpha(-\tan\beta)}\quad(\because \tan(-\beta)=-\tan\beta)$$

$$\therefore \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Example 1: Find $\tan 15°$ exactly.

Solution: We rewrite $15°$ as $45°-30°$ and using the identity

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$\tan 15^o=\tan(45^o-30^o)=\frac{\tan 45^o-\tan 30^o}{1+\tan 45^o\tan 30^o}=\frac{1-\dfrac{1}{\sqrt3}}{1+1\cdot\dfrac{1}{\sqrt3}}=\frac{\sqrt3-1}{\sqrt3+1}=\frac{3-\sqrt3}{3+\sqrt3}$$

Example 2: Find the exact value of: $\sin 42°\cos 12°-\cos 42°\sin 12°$.

Solution: Using the identity $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$$\sin 42°\cos 12°-\cos 42°\sin 12°=\sin(42°-12°)=\sin 30°=\frac{1}{2}.$$

Example 3: Given $\sin\alpha = \frac{12}{13}$ and $\cos\beta = \frac{3}{5}$, where α and β are in the first quadrant. Find in which quadrant does $(\alpha+\beta)$ lie.

Solution: Given that α, β are both in the first quadrant. Since cosine is positive in the first quadrant and negative in the second quadrant, therefore, $\cos(\alpha+\beta)$ will decide the quadrant in which $(\alpha+\beta)$ lies?

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad (1)$$

As $\cos^2\alpha = 1 - \sin^2\alpha$, putting $\sin^2\alpha = \left(\frac{12}{13}\right)^2 = \frac{144}{169}$

$$\cos^2\alpha = 1 - \frac{144}{169} = \frac{169-144}{169} = \frac{25}{169}$$

$\cos\alpha = \pm\frac{5}{13}$. But $\cos\alpha$ is +ve in the 1st quadrant,

$$\therefore \cos\alpha = +\frac{5}{13}$$

As $\sin^2\beta = 1 - \cos^2\beta$, putting in it $\cos^2\beta = \frac{9}{25}$

$$\sin^2\beta = 1 - \frac{9}{25} = \frac{25-9}{25} = \frac{16}{25} = \pm\frac{4}{5}$$

But $\sin\beta$ is +ve in the 1st quadrant,

$$\therefore \sin\beta = \frac{4}{5}$$

Putting values of $\sin\alpha$, $\cos\alpha$, $\sin\beta$ and $\cos\beta$ in (1)

$$\cos(\alpha+\beta) = \left(\frac{5}{13}\right)\left(\frac{3}{5}\right) - \left(\frac{12}{13}\right)\left(\frac{4}{5}\right) = \left(\frac{15}{65}\right) - \left(\frac{48}{65}\right) = \frac{15-48}{65} = \frac{-33}{65}$$

Since $\cos(\alpha+\beta)$ is negative, it follows that $(\alpha+\beta)$ is in the second quadrant.

10.2 Trigonometric ratios of allied angles

10.2.1 The angles of measure $\frac{\pi}{2}\pm\theta, \pi\pm\theta, \frac{3\pi}{2}\pm\theta, 2\pi\pm\theta$ are called **allied angles.**

Thus the angles which are connected with basic angles of measure θ by a right angle or its multiple are known as allied angles.

10.2.2 Derivation of trigonometric ratios of allied angles

All the following trigonometric ratios of allied angles can be derived from the fundamental theorem of trigonometry and thus has been left for the students as an exercise.

i. $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$, $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$

ii. $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$

iii. $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$, $\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$

iv. $\sin(\pi-\theta) = \sin\theta$, $\cos(\pi-\theta) = -\cos\theta$

v. $\sin(\pi+\theta) = -\sin\theta$, $\cos(\pi+\theta) = -\cos\theta$

vi. $\tan(\pi-\theta) = -\tan\theta$, $\tan(\pi+\theta) = \tan\theta$

vii. $\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$, $\cos\left(\frac{3\pi}{2}-\theta\right) = -\sin\theta$

viii. $\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$, $\cos\left(\frac{3\pi}{2}+\theta\right) = \sin\theta$

ix. $\tan\left(\frac{3\pi}{2}+\theta\right) = \cot\theta$, $\tan\left(\frac{3\pi}{2}+\theta\right) = -\cot\theta$

x. $\sin(2\pi-\theta) = -\sin\theta$, $\cos(2\pi-\theta) = \cos\theta$

xi. $\sin(2\pi+\theta) = \sin\theta$, $\cos(2\pi+\theta) = \cos\theta$

xii. $\tan(2\pi-\theta) = -\tan\theta$, $\tan(2\pi+\theta) = \tan\theta$

Note: 1. The above results also apply to the reciprocals of sine, cosine and tangent. These results are to be applied frequently in the study of trigonometry.

2. They can be obtained by using the following two-steps procedure:

a)

First quadrant	(0 , π/2)	All are +ve
Second quadrant	(π/2, π)	sin is +ve
Third quadrant	(π , 3π/2)	tan is +ve
Fourth quadrant	(3π/2, 2π)	cos is +ve

b) If we have $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ in the formula, the formula changes sine to cosine and cosine to sine, tangent to cotangent and cotangent to tangent, secant to cosecant and cosecant to secant. If we have π or 2π in the formula, the function does not change.

Example 4: Simplify each expression, given that $0 < x < \pi/2$.

(i) $\sin(\pi/2 + x)$ (ii) $\cos(\pi/2 + x)$ (iii) $\tan(3\pi/2 + x)$

(iv) $\cot(2\pi - x)$ (v) $\sin(\pi + x)$ (vi) $\cos(2\pi + x)$

Solution: (i) $(\pi/2 + x)$ is in the second quadrant, so $\sin(\pi/2 + x) = \cos x$

(ii) $(\pi/2 + x)$ is in the second quadrant, so $\cos(\pi/2 + x) = -\sin x$

(iii) $(3\pi/2 + x)$ is in the fourth quadrant, so $\tan(3\pi/2 + x) = -\cot x$

(iv) $(2\pi - x)$ is in the fourth quadrant, so $\cot(2\pi - x) = -\sin x$

(v) $(\pi + x)$ is in the third quadrant, so $\sin(\pi + x) = -\cot x$

(vi) $(2\pi + x)$ is in the first quadrant, so $\cos(2\pi + x) = \cos x$

Example 5: Simplify $\dfrac{\cos(90°+x)+\sin(270°-x)+\sin(180°-x)}{\cos(-x)-\cos(360°-x)+\sin(90°+x)}$

Solution: $\dfrac{\cos(90°+x)+\sin(270°-x)+\sin(180°-x)}{\cos(-x)-\cos(360°-x)+\sin(90°+x)}$

$$=\frac{-\sin x-\cos x+\sin x}{\cos x-\cos x+\cos x}=\frac{-\cos x}{\cos x}=-1$$

Example 6: If α, β, γ are the angles of Δ ABC. Prove that

i) $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

ii) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=-1$

Solution: As α, β, γ are the angles of ΔABC $\therefore \alpha + \beta + \gamma = 180^0$

i) $\alpha + \beta = 180^0 - \gamma$

$\tan(\alpha + \beta) = \tan(180^0 - \gamma)$ $\Rightarrow \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \tan\gamma$

$\Rightarrow \tan\alpha + \tan\beta = -\tan\gamma + \tan\alpha\tan\beta\tan\gamma$

$\therefore \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

ii) As $\alpha + \beta + \gamma = 180^0 \Rightarrow \frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2} = 90^0$

So $\frac{\alpha}{2}+\frac{\beta}{2} = 90^0 - \frac{\gamma}{2}$ $\therefore \tan(\frac{\alpha}{2}+\frac{\beta}{2}) = \tan\left(90^0 - \frac{\gamma}{2}\right)$

$$\frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}=\cot\frac{\gamma}{2}=\frac{1}{\tan\frac{\gamma}{2}} \Rightarrow \tan\frac{\alpha}{2}\tan\frac{\gamma}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$$

10.2.3 Writing $a\sin\theta + b\cos\theta$ in the form $r\sin(\theta+\phi)$ where $a = r\cos\phi$ and $b = r\sin\phi$

Writing $a\sin\theta + b\cos\theta$ in the Form $r\sin(\theta+\phi)$.

Let P(a, b) be a coordinate point in the plane and let θ be the angle with initial side x-axis and terminal side the ray $\overrightarrow{OP}$ as shown in Figure 10.2.

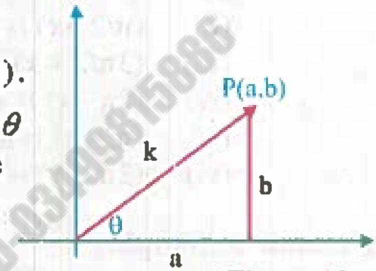


Figure 10.2

We can express $a\sin\theta + b\cos\theta$ in the form $r\sin(\theta+\phi)$ where $r=\sqrt{a^2+b^2}$ and ϕ is given by the equations $r\cos\phi = a$ and $r\sin\phi = b$.

The method is explained through the following example.

Example 7: Express $5\sin\theta + 12\cos\theta$ in the form $r\sin(\theta+\phi)$, where the terminal side of the angle of measure ϕ is in the 1st quadrant.

Solution: Identifying $5\sin\theta + 12\cos\theta$ with $r\sin(\theta+\phi)$ gives

$$5\sin\theta + 12\cos\theta = r\cos\phi\sin\theta + r\sin\phi\cos\theta \quad (1)$$

so $\quad 5 = r\cos\phi$ and $12 = r\sin\phi$

$\therefore \quad r=\sqrt{a^2+b^2} = \sqrt{(5)^2+(12)^2} = \sqrt{25+144} = \sqrt{169} = 13$

and $\quad r\cos\phi = 5 \Rightarrow 13\cos\phi = 5 \Rightarrow \cos\phi = \frac{5}{13}$,

$\quad r\sin\phi = 12 \Rightarrow 13\sin\phi = 12 \Rightarrow \sin\phi = \frac{12}{13}$.

Thus, from (1) we get

$$5\sin\theta + 12\cos\theta = 13\left(\frac{5}{13}\sin\theta+\frac{12}{13}\cos\theta\right)$$

$$= 13\left(\sin\theta\frac{5}{13}+\cos\theta\frac{12}{13}\right) = r(\sin\theta\cos\phi+\cos\theta\sin\phi)$$

$$= r\sin(\theta+\phi) \quad \text{where } \sin\phi = \frac{12}{13}, \cos\phi=\frac{5}{13} \text{ and } r = 13$$

EXERCISE 10.1

1. Write each of the following as a trigonometric function of a single angle.

(i) $\sin 37^\circ \cos 22^\circ + \cos 37^\circ \sin 22^\circ$ (ii) $\cos 83^\circ \cos 53^\circ + \sin 83^\circ \sin 53^\circ$

(iii) $\cos 19^\circ \cos 5^\circ - \sin 19^\circ \sin 5^\circ$ (iv) $\sin 40^\circ \cos 15^\circ - \cos 40^\circ \sin 15^\circ$

(v) $\dfrac{\tan 20^\circ + \tan 32^\circ}{1 - \tan 20^\circ \tan 32^\circ}$ (vi) $\dfrac{\tan 35^\circ - \tan 12^\circ}{1 + \tan 35^\circ \tan 12^\circ}$

2. Evaluate each of the following exactly.

(i) $\sin \dfrac{\pi}{12}$ (ii) $\tan 75^\circ$ (iii) $\tan 105^\circ$ (iv) $\tan \dfrac{5\pi}{12}$ (v) $\cos 15^\circ$ (vi) $\sin \dfrac{7\pi}{12}$

3. If $\sin u = \dfrac{3}{5}$ and $\sin v = \dfrac{4}{5}$ and u and v are between 0 and $\dfrac{\pi}{2}$, evaluate each of the following exactly.

(i) $\cos(u+v)$ (ii) $\tan(u-v)$ (iii) $\sin(u-v)$ (iv) $\cos(u-v)$

4. If $\sin \alpha = -\dfrac{4}{5}$ and $\cos \beta = -\dfrac{12}{13}$, α in Quadrant III and β in Quadrant II, find the exact value of:

(i) $\sin(\alpha - \beta)$ (ii) $\cos(\alpha + \beta)$ (iii) $\tan(\alpha + \beta)$

5. If $\tan \alpha = \dfrac{3}{4}$, $\sec \beta = \dfrac{13}{5}$, and neither the terminal side of the angle of measure α nor β in the first quadrant, then find:

(i) $\sin(\alpha + \beta)$ (ii) $\cos(\alpha + \beta)$ (iii) $\tan(\alpha + \beta)$

6. Show that:

(i) $\cos \alpha = 2\cos^2 \dfrac{\alpha}{2} - 1 = 1 - 2\sin^2 \dfrac{\alpha}{2}$

(ii) $\sin(\alpha + \beta)\sin(\alpha - \beta) = \cos^2 \beta - \cos^2 \alpha$

7. Show that: (i) $\cot(\alpha + \beta) = \dfrac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta}$ (ii) $\dfrac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta} = \tan \alpha + \tan \beta$

8. Prove that: (i) $\tan\left(\dfrac{\pi}{4} + \theta\right) = \dfrac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta}$ (ii) $\tan\left(\dfrac{\pi}{4} - \theta\right) = \dfrac{1 - \tan \theta}{1 + \tan \theta}$

(iii) $\dfrac{\tan(\alpha + \beta)}{\cot(\alpha - \beta)} = \dfrac{\tan^2 \alpha - \tan^2 \beta}{1 - \tan^2 \alpha \tan^2 \beta}$ (iv) $\dfrac{1 - \tan \theta \tan \phi}{1 + \tan \theta \tan \phi} = \dfrac{\cos(\theta + \phi)}{\cos(\theta - \phi)}$

9. Prove that: $\dfrac{\sin\theta}{\sec 4\theta}+\dfrac{\cos\theta}{\operatorname{cosec}4\theta}=\sin 5\theta$

10. Show that: $\dfrac{\sin(180^\circ-\alpha)\cos(270^\circ-\alpha)}{\sin(180^\circ+\alpha)\cos(270^\circ+\alpha)}=1$

11. If α, β, γ are the angles of a triangle ABC, show that

$$\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$$

12. If $\alpha+\beta+\gamma=180^0$, show that $\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha=1$

13. Express each of the following in the form $r\sin(\theta+\phi)$ where terminal ray of θ and ϕ are in the first quadrant.

(i) $4\sin\theta+3\cos\theta$. (ii) $15\sin\theta+8\cos\theta$.

(iii) $2\sin\theta-5\cos\theta$. (iv) $\sin\theta+\cos\theta$.

10.3 Double, Half and Triple Angle Identities

In this section we derive formulae/identities for $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$ for $\sin\frac{1}{2}\theta$, $\cos\frac{1}{2}\theta$ and $\tan\frac{1}{2}\theta$ and for $\sin 3\theta$, $\cos 3\theta$ and $\tan 3\theta$ called double angle, half angle and triple angle formulae respectively.

10.3.1 Double Angle Identities

We know that. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ (1)

and $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ (2) Putting $\beta=\alpha$ in (1).

$\sin(\alpha+\alpha)=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha$

$\sin 2\alpha=2\sin\alpha\cos\alpha$ (3)

Now putting $\beta=\alpha$ in (2)

$\cos(\alpha+\alpha)=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha$

$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$ (4)

Putting $\cos^2\alpha=1-\sin^2\alpha$ in (4) $(\because \sin^2\alpha+\cos^2\alpha=1)$

$\cos 2\alpha=1-\sin^2\alpha-\sin^2\alpha$

$\cos 2\alpha=1-2\sin^2\alpha$ (5)

Now putting $\sin^2\alpha = 1-\cos^2\alpha$ in (4)

$$\cos 2\alpha = \cos^2\alpha - (1-\cos^2\alpha)$$

$$\cos 2\alpha = \cos^2 - 1 + \cos^2\alpha$$

$$\cos 2\alpha = 2\cos^2\alpha - 1 \qquad (6)$$

We also know that $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ Putting $\beta = \alpha$

$$\tan(\alpha+\alpha) = \frac{\tan\alpha+\tan\alpha}{1-\tan\alpha.\tan\alpha}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} \qquad (7)$$

Example 8: Given that $\tan\theta = -\dfrac{3}{4}$ *and* θ is in the quadrant II, find each of the following.

i) $\cos 2\theta$ *ii)* $\cos 2\theta$

iii) $\tan 2\theta$ iv) The quadrant in which 2θ lies

Soultion: By drawing a refernce triangle as shown,

we find that

$$\sin\theta = \frac{3}{5} \qquad \text{And} \qquad \cos\theta = \frac{4}{5}$$

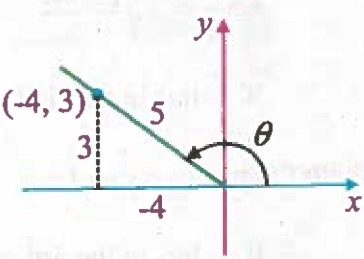


Figure 10.3

Thus we have the following.

i) $\sin 2\theta = 2\sin\theta\cos\theta = 2.\dfrac{3}{5}.\left(-\dfrac{4}{5}\right) = -\dfrac{24}{25}$

ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(-\dfrac{4}{5}\right)^2 - \left(\dfrac{3}{5}\right)^2 = \dfrac{16}{25} - \dfrac{9}{25} = \dfrac{7}{25}$

iii) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2\left(-\dfrac{3}{4}\right)}{1-\left(-\dfrac{3}{4}\right)} = \dfrac{-\dfrac{3}{2}}{1-\dfrac{9}{16}} = -\dfrac{3}{2}.\dfrac{16}{7} = -\dfrac{24}{7}$

Note that $\tan 2\theta$ could have been found more easily in this case simply as following:

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}.$$

iv) Since $\sin 2\theta$ is negative and $\cos 2\theta$ is positive, we know that 2θ is in quadrant IV.

10.3.2 Half Angle Identities

We have $\cos 2\alpha = 1 - 2\sin^2\alpha \Rightarrow 2\sin^2\alpha = 1 - \cos 2\alpha$

$$\Rightarrow \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$\sin\alpha = \pm\sqrt{\frac{1-\cos 2\alpha}{2}} \qquad (8)$$

Now putting $\alpha = \frac{\theta}{2}$ in (8).

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\left(2.\frac{\theta}{2}\right)}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad (9)$$

If $\frac{\theta}{2}$ lies in the first or second quadrant then we will write the identity (9) with the positive sign i.e. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$

If $\frac{\theta}{2}$ lies in the 3rd or 4th quadrant, we will write the identity (9) with the negative sign i.e.

$$\sin\frac{\theta}{2} = -\sqrt{\frac{1-\cos\theta}{2}}$$

Also we know that

$$\cos 2\alpha = 2\cos^2\alpha - 1 \Rightarrow 2\cos^2\alpha = 1 + \cos 2\alpha$$

$\Rightarrow \cos^2\alpha = \frac{1+\cos 2\alpha}{2} \Rightarrow \cos\alpha = \pm\sqrt{\frac{1+\cos 2\alpha}{2}}$. Putting $\alpha = \frac{\theta}{2}$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos(2\cdot\frac{\theta}{2})}{2}}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}} \qquad (10)$$

From (9) and (10), we have $\tan\frac{\theta}{2}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

$$\tan\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \qquad (11)$$

Example 9: Find tan (π/8) exactly.

Solution:

$$\tan\frac{\pi}{8}=\tan\left(\frac{\frac{\pi}{4}}{2}\right)=\sqrt{\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}}=\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}}=\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}=\sqrt{\frac{\sqrt{2}(\sqrt{2}-1)}{\sqrt{2}(\sqrt{2}+1)}}=\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$$

$$=\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}}}=\sqrt{\frac{4-2\sqrt{2}-2\sqrt{2}+2}{4-2}}=\sqrt{\frac{6-4\sqrt{2}}{2}}=\sqrt{3-2\sqrt{2}}$$

The identities that we have developed are also useful for simplifying trigonometric expressions.

Example 10: Simplify each of the following.

a) $\dfrac{\sin x\cos x}{\frac{1}{2}\cos 2x}$ b) $2\sin^2\frac{x}{2}+\cos x$

Solution: *a)* $\dfrac{\sin x\cos x}{\frac{1}{2}\cos 2x}=\dfrac{2}{2}\cdot\dfrac{\sin x\cos x}{\frac{1}{2}\cos 2x}=\dfrac{2\sin x\cos x}{\cos 2x}$

$$=\frac{\sin 2x}{\cos 2x}=\tan 2x \qquad (\text{using} \sin 2x=2\sin x\cos x)$$

b) $2\sin^2\frac{x}{2}+\cos x=2\left(\frac{1-\cos x}{2}\right)+\cos x\left(\text{using} \sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}, \textit{ or } \sin^2\frac{x}{2}=\frac{1-\cos x}{2}\right)$

$$=1-\cos x+\cos x=1.$$

10.3.3. Triple Angle Identities

We have $\sin 3\alpha = \sin(2\alpha + \alpha) = \sin 2\alpha \cos\alpha + \cos 2\alpha \sin\alpha$

$$= 2\sin\alpha\cos\alpha\cos\alpha + (1 - 2\sin^2\alpha)\sin\alpha \quad \text{(By (3) and (5))}$$

$$= 2\sin\alpha\cos^2\alpha + \sin\alpha - 2\sin^3\alpha$$

$$= 2\sin\alpha(1 - \sin^2\alpha) + \sin\alpha - 2\sin^3\alpha \quad (\because \sin^2 + \cos^2\alpha = 1)$$

$$= 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

$$\therefore \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \qquad (12)$$

$$\cos 3\alpha = \cos(2\alpha + \alpha)$$

$$= \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha$$

$$= (2\cos^2\alpha - 1)\cos\alpha - 2\sin\alpha\cos\alpha\sin\alpha \quad \text{(by (3) and (6))}$$

$$= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha$$

$$= 2\cos^3\alpha - \cos\alpha - 2(1 - \cos^2\alpha)\cos\alpha \quad (\because \sin^2 + \cos^2\alpha = 1)$$

$$= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\therefore \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha \qquad (13)$$

$$\tan 3\alpha = \tan(2\alpha + \alpha) = \frac{\tan 2\alpha + \tan\alpha}{1 - \tan 2\alpha\tan\alpha} = \frac{\frac{2\tan\alpha}{1 - \tan^2\alpha} + \tan\alpha}{1 - \frac{2\tan\alpha}{1 - \tan^2\alpha}.\tan\alpha} \quad \text{(By (7))}$$

$$= \frac{\frac{2\tan\alpha + \tan\alpha(1 - \tan^2\alpha)}{1 - \tan^2\alpha}}{\frac{1 - \tan^2\alpha - 2\tan^2\alpha}{1 - \tan^2\alpha}} = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

$$\therefore \tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} \qquad (14)$$

Example 11: Prove the identity $\dfrac{\sin 2x}{\sin x} - \dfrac{\cos 2x}{\cos x} = \sec x.$

Solution: $\dfrac{\sin 2x}{\sin x} - \dfrac{\cos 2x}{\cos x} = \dfrac{2\sin x\cos x}{\sin x} - \dfrac{\cos^2 x - \sin^2 x}{\cos x}$ (using double – angle idnetities)

$$= 2\cos x - \frac{\cos^2 x - \sin^2 x}{\cos x} \quad \text{(simplifying)}$$

$$= \frac{2\cos^2 x - \cos^2 x + \sin^2 x}{\cos x} \quad \text{(taking LCM and simplifying)}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos x} = \frac{1}{\cos x} = \sec x$$

We started with the left side and obtained the right side, so the proof is complete.

Example 12: Prove the identity

$$\sin^2 x \tan^2 x = \tan^2 x - \sin^2 x.$$

Solution: For this proof, we are going to work with each side separately. We try to obtain the same expression on each side.

$$\sin^2 x \tan^2 x = \sin^2 x\left(\frac{\sin^2 x}{\cos^2 x}\right) = \frac{\sin^4 x}{\cos^2 x} \ldots\ldots(1)$$

$$\tan^2 x - \sin^2 x = \frac{\sin^2 x}{\cos^2 x} - \sin^2 x \qquad \left(\because \tan x = \frac{\sin x}{\cos x}\right)$$

$$= \frac{\sin^2 x - \sin^2 x\cos^2 x}{\cos^2 x} \quad \text{(Taking LCM)}$$

$$= \frac{\sin^2 x(1 - \cos^2 x)}{\cos^2 x} \quad \text{(Factoring)}$$

$$= \frac{\sin^2 x \sin^2 x}{\cos^2 x} \quad \text{(Recalling the identity } 1 - \cos^2 x = \sin^2 x\text{)}$$

$$= \frac{\sin^4 x}{\cos^2 x} \ldots\ldots\ldots(2)$$

We have obtained the same expression from each side, so the proof is complete.

Example 13: Find the exact value of cos 105°.

Solution: Because $105° = \frac{1}{2}(210°)$ we can find cos 105° by using the half-angle identity for cos $\alpha/2$ with $\alpha = 210°$. The angle $\alpha/2 = 105°$ lies in Quadrant II, and the cosine function is negative in Quadrant II. Thus $\cos 105° < 0$, and we must select the minus sign that precedes the radical in $\cos\alpha = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$ to produce the correct result.

$$\cos 105^\circ = -\sqrt{\frac{1+\cos 210^\circ}{2}} = -\sqrt{\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}} = -\sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{3}}{4}}$$

Example 14: Show that,

$$\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

Solution: L.H.S $= \sin^4\theta = (\sin^2\theta)^2$

$$= \left(\frac{1-\cos 2\theta}{2}\right)^2 \qquad \left(\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$= \frac{1-2\cos 2\theta+\cos^2 2\theta}{4} = \frac{1}{4}[1-2\cos 2\theta+\cos^2 2\theta]$$

$$= \frac{1}{4}\left[1-2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right] \qquad \left(\because \cos^2 2\theta = \frac{1+\cos 4\theta}{2}\right)$$

$$= \frac{1}{4}\left[\frac{2-4\cos 2\theta+1+\cos 4\theta}{2}\right] = \frac{1}{8}[3-4\cos 2\theta+\cos 4\theta]$$

$$= \frac{3}{8} - \frac{4}{8}\cos 2\theta + \frac{1}{8}\cos 4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta = \text{R.H.S.}$$

$$\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

Example 15: Prove the following identities.

(i) $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$ (ii) $\sin 4\theta = 8\sin\theta\cos^3\theta - 4\sin\theta\cos\theta$

Solution: (i) $R.H.S = \dfrac{2\tan\theta}{1+\tan^2\theta} = 2\cdot\dfrac{\tan\theta}{\sec^2\theta}$ $\qquad (\because \tan^2 a + 1 = \sec^2 a)$

$= 2\cdot\tan\theta\cos^2\theta$ $\qquad \left(\because \cos a = \dfrac{1}{\sec a}\right)$

$$= 2\cdot\frac{\sin\theta}{\cos\theta}\cos^2\theta$$

$$= 2\sin\theta\cos\theta$$

$$= \sin 2\theta$$

(ii) $\sin 4\theta = 8\sin\theta\cos^3\theta - 4\sin\theta\cos\theta$

$$\text{L.H.S} = \sin 4\theta = \sin\left[2(2\theta)\right]$$

$$= 2\sin 2\theta\cos 2\theta \quad (\text{Use } \sin 2a = 2\sin a\cos a, \text{ with } a = 2\theta)$$

$$= 2(2\sin\theta\cos\theta)(2\cos^2\theta - 1) = 4\sin\theta\cos\theta(2\cos^2\theta - 1)$$

$$= 8\sin\theta\cos^3\theta - 4\sin\theta\cos\theta = R.H.S$$

EXERCISE 10.2

1. Find the values of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$, given $\tan\theta = -\frac{1}{5}$, θ in quadrant II.

2. If $\sin\theta = \frac{5}{13}$ and terminal ray of θ is in the second quadrant, then find

   (i) $\sin 2\theta$ (ii) $\cos 2\theta$ (iii) $\tan 2\theta$

3. If $\sin\theta = \frac{4}{5}$ and terminal ray of θ is in the second quadrant, then find

   (i) $\sin 2\theta$ (ii) $\cos\frac{\theta}{2}$

4. If $\cos\theta = -\frac{3}{7}$ and terminal ray of θ is in 3rd quadrant, then find $\sin\frac{\theta}{2}$.

5. Use double angle identities to evaluate exactly.

   (i) $\sin\frac{2\pi}{3}$ (ii) $\cos\frac{2\pi}{3}$

6. Use the half-angle identities to evaluate exactly.

   (i) $\cos 15^\circ$ (ii) $\tan 67.5^\circ$ (iii) $\sin 112.5^\circ$

   (iv) $\cos\frac{\pi}{8}$ (v) $\tan 75^\circ$ (vi) $\sin\frac{5\pi}{12}$

7. Prove the following identities:

(i) $\cos^4\theta - \sin^4\theta = \dfrac{1}{\sec 2\theta}$

(ii) $\tan\dfrac{\theta}{2} + \cot\dfrac{\theta}{2} = \dfrac{2}{\sin\theta}$

(iii) $\dfrac{1+\cos 2\theta}{1+\cos 2\theta} = \cot^2\theta$

(iv) $\operatorname{cosec}2\theta - \cot 2\theta = \tan\theta$

(v) $\dfrac{\sin 3\beta}{\sin\beta} - \dfrac{\cos 3\beta}{\cos\beta} = 2$

(vi) $\dfrac{\sin 3\theta}{\cos\theta} + \dfrac{\cos 3\theta}{\sin\theta} = 2\cot 2\theta$

(vii) $\dfrac{\cos^3\theta - \sin^3\theta}{\cos\theta - \sin\theta} = \dfrac{2+\sin 2\theta}{2}$

(viii) $\dfrac{2\cos^3\theta}{1-\sin\theta} = 2\cos\theta + \sin 2\theta$

(ix) $\cot 2\theta = \dfrac{1}{2}\left(\cot\theta - \dfrac{1}{\cot\theta}\right)$

(x) $\dfrac{\sin\alpha + \cos\alpha}{\cos\alpha - \sin\alpha} + \dfrac{\sin\alpha - \cos\alpha}{\cos\alpha + \sin\alpha} = 2\tan 2\alpha$

(xi) $\tan\dfrac{\alpha}{2} = \dfrac{\sin\alpha}{1+\cos\alpha}$

(xii) $\dfrac{\operatorname{cosec}\theta - \cot\theta}{1+\cos\theta} = \operatorname{cosec}\theta\tan^2\dfrac{\theta}{2}$

(xiii) $\cos^2\dfrac{\theta}{2} = \dfrac{1-\cos^2\theta}{2-2\cos\theta}$

(xiv) $\dfrac{1-\tan^2\dfrac{\alpha}{2}}{1+\tan^2\dfrac{\alpha}{2}} = \cos\alpha$

(xv) $\sin 2\theta - 4\sin^3\theta\cos\theta = \sin 2\theta\cos 2\theta$

8. Write $\cos^4\theta$ in terms of the first power of one or more cosine functions.

9. Prove the following identities:

(i) $\sin 4\theta = 8\sin\theta\cos^3\theta - 4\sin\theta\cos\theta$

(ii) $\cot 4\theta = \dfrac{1-\tan^2 2\theta}{2\tan 2\theta}$

(iii) $\cot 3\theta = \dfrac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$

10.4 Sum, Difference and Product of sine and cosine

10.4.1 Converting Product to Sums or Differences

We know that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad (1)$$

and

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \qquad (2)$$

Adding (1) and (2) we get

$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$

$\therefore 2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$ (3)

Now Subtracting (2) from (1)

$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta$

$\therefore 2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$ (4)

We also know that;

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ (5)

and $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ (6)

Adding (5) and (6) we have,

$\therefore 2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$ (7)

Subtracting (6) from (5) we get,

$-2\sin\alpha\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$

$\therefore 2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$ (8)

So, by converting products into sums or differences we get the following four identities:

$$2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$$
$$2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$$
$$2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$$
$$2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$$

These identities are usually called the **Product–to–Sum formulae.**

Example 16: Write the product $2\sin 5\theta\cos 3\theta$ as a sum or difference of sine and cosine.

Solution: Using the identity $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$

We have,

$$2\sin 5\theta\cos 3\theta=\sin(5\theta+3\theta)+\sin(5\theta-3\theta)=\sin 8\theta+\sin 2\theta$$

Example 17: Express $\sin 10\theta\cos 4\theta$ as a sum or difference.

Solution: Using the identity $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$

We have, $\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

$$\sin 10\theta\cos 4\theta=\frac{1}{2}[\sin(10\theta+4\theta)+\sin(10\theta-4\theta)]=\frac{1}{2}(\sin 14\theta+\sin 6\theta)$$

Example 18: Write the product 2 cos 45° cos 15° as a sum or difference.

Solution: Using the identity $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

We have,

$$\therefore \quad 2\cos 45^\circ \cos 15^\circ = \cos(45^\circ+15^\circ) + \cos(45^\circ-15^\circ)$$
$$= \cos 60^\circ + \cos 30^\circ$$

10.4.2 Converting Sums or Differences to Products

Let $\alpha+\beta = \theta$ .... (1)

$\alpha-\beta = \phi$ .... (2)

Adding (1) and (2), we have

$$\alpha = \frac{\theta+\phi}{2}$$

Subtracting (2) from (1), we have

$$\beta = \frac{\theta-\phi}{2}$$

Substituting $\alpha = \frac{\theta+\phi}{2}$ and $\beta = \frac{\theta-\phi}{2}$ in the four identities of section 10.4.1, we get

$$\sin\theta + \sin\phi = 2\sin\frac{\theta+\phi}{2}.\cos\frac{\theta-\phi}{2}$$
$$\sin\theta - \sin\phi = 2\cos\frac{\theta+\phi}{2}.\sin\frac{\theta-\phi}{2}$$
$$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}.\cos\frac{\theta-\phi}{2}$$
$$\cos\theta - \cos\phi = -2\sin\frac{\theta+\phi}{2}.\sin\frac{\theta-\phi}{2}$$

These identities are usually called the **sum-to-product formulae.**

Example 19: Convert the sum sin16° + sin12° into product.

Solution: We know that, $\sin\theta + \sin\phi = 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$

$\therefore \quad \sin 16° + \sin 12° = 2\sin\frac{16°+12°}{2}\cos\frac{16°-12°}{2} = 2\sin\frac{28°}{2}\cos\frac{4°}{2}$

$= 2\sin 14° \cos 4°$

Example 20: Express cos4θ – cos2θ as a product.

Solution: We have $\cos\theta - \cos\phi = -2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$

$$\therefore \cos4\theta - \cos2\theta = -2\sin\frac{4\theta+2\theta}{2}\sin\frac{4\theta-2\theta}{2}$$

$$= -2\sin\frac{6\theta}{2}\sin\frac{2\theta}{2} = -2\sin3\theta\sin\theta$$

Example 21: Show that $\dfrac{\cos\alpha-\cos\beta}{\sin\alpha+\sin\beta} = -\tan\frac{1}{2}(\alpha-\beta)$

Solution: $\text{L.H.S} = \dfrac{\cos\alpha-\cos\beta}{\sin\alpha+\sin\beta} = \dfrac{-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}} = \dfrac{-\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}} = -\tan\frac{\alpha-\beta}{2}$

$$= -\tan\frac{1}{2}(\alpha-\beta) = \text{R.H.S}$$

Example 22: Show that $\sin5\theta + 2\sin3\theta + \sin\theta = 4\sin3\theta\cos^2\theta$

Solution:

$\text{L.H.S} = \sin5\theta + 2\sin3\theta + \sin\theta = (\sin5\theta + \sin3\theta) + (\sin3\theta + \sin\theta)$

$$= 2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right) + 2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)$$

$$= 2\sin\frac{8\theta}{2}\cos\frac{2\theta}{2} + 2\sin\frac{4\theta}{2}\cos\frac{2\theta}{2} = 2\sin4\theta\cos\theta + 2\sin2\theta\cos\theta$$

$$= 2\cos\theta(\sin4\theta + \sin2\theta) = 2\cos\theta\left[2\sin\left(\frac{4\theta+2\theta}{2}\right)\cos\left(\frac{4\theta-2\theta}{2}\right)\right]$$

$$= 2\cos\theta(2\sin3\theta\cos\theta) = 4\sin3\theta\cos^2\theta = \text{R.H.S.}$$

Example 23: Show that $\left(\dfrac{\sin3\theta+\sin\theta}{\sin3\theta-\sin\theta}\right)\left(\dfrac{\cos\theta+\cos\theta}{\cos3\theta-\cos\theta}\right) = -\cot^2\theta$

Solution:

$$L.H.S = \left(\frac{\sin3\theta+\sin\theta}{\sin3\theta-\sin\theta}\right)\left(\frac{\cos3\theta+\cos\theta}{\cos3\theta-\cos\theta}\right) = \left(\frac{2\sin2\theta\cos\theta}{2\sin\theta\cos2\theta}\right)\left(\frac{2\cos2\theta\cos\theta}{-2\sin2\theta\sin\theta}\right)$$

$$=\left(\frac{2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)}{2\cos\left(\frac{3\theta+\theta}{2}\right)\sin\left(\frac{3\theta-\theta}{2}\right)}\right)\left(\frac{2\cos\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)}{2\sin\left(\frac{3\theta+\theta}{2}\right)\sin\left(\frac{3\theta-\theta}{2}\right)}\right)=-\frac{\cos^2\theta}{\sin^2\theta}=-\cot^2\theta.$$

Example 24: Show that $\cos20^0\cos40^0\cos80^0 = 1/8$

Solution:

L.H.S. $= \cos20^0\cos40^0\cos80^0 = \frac{1}{4}[4\cos20^0\cos40^0\cos80^0]$

$= \frac{1}{4}[(2\cos40^0\cos20^0)\ 2\cos80^0] = \frac{1}{4}[(\cos60^0+\cos20^0)\ 2\cos80^0]$

$= \frac{1}{4}[(1/2+\cos20^0)\ 2\cos80^0] = \frac{1}{4}[\cos80^0 + 2\cos80^0\cos20^0]$

$= \frac{1}{4}[\cos80^0+\cos100^0+\cos60^0] = \frac{1}{4}[\cos80^0+\cos(180^0-80^0)+\cos60^0]$

$= \frac{1}{4}[\cos80^0-\cos80^0+\frac{1}{2}]$ $\quad\because \cos(180^0-\theta) = -\cos\theta$

$= \frac{1}{4}[1/2] = 1/8 =$ R.H.S

EXERCISE 10.3

1. Express the following products as sums or differences.

 (i) $2\sin6x\sin x$ (ii) $\sin55^o\cos123^o$

 (iii) $\sin\frac{A+B}{2}\cos\frac{A-B}{2}$ (iv) $\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$

2. Convert the following sums or differences to products:

 (i) $\sin37^o+\sin43^o$ (ii) $\cos36^o-\cos82^o$

 (iii) $\sin\frac{P+Q}{2}-\sin\frac{P-Q}{2}$ (iv) $\cos\frac{A+B}{2}+\cos\frac{A-B}{2}$

3. Prove the following.

 (i) $\frac{\cos75^o+\cos15^o}{\sin75^o-\sin15^o}=\sqrt{3}$ (ii) $\frac{\sin135^0-\cos120^0}{\sin135^0+\cos120^0}=3+2\sqrt{2}$

4. Prove the following identities:

 (i) $\frac{\sin\alpha+\sin9\alpha}{\cos\alpha+\cos9\alpha}=\tan5\alpha$ (ii) $\frac{\cos\beta+\cos3\beta+\cos5\beta}{\sin\beta+\sin3\beta+\sin5\beta}=\cot3\beta.$

 (iii) $\sin2\theta+\sin4\theta+\sin6\theta=4\cos\theta\cos2\theta\sin3\theta$

 (iv) $\sin5\theta+\sin\theta+2\sin3\theta=4\sin3\theta\cos^2\theta$

 (v) $\sin3\theta+\sin5\theta+\sin7\theta+\sin9\theta=4\cos\theta\sin6\theta\cos2\theta$

(vi) $\cos\beta + \cos 2\beta + \cos 5\beta = \cos 2\beta\,(1+2\cos 3\beta)$

5. Prove that (i) $\cos 20^0 \cos 40^0 \cos 60^0 \cos 80^0 = \dfrac{1}{16}$

(ii) $\sin\dfrac{\pi}{9}\sin\dfrac{2\pi}{9}\sin\dfrac{\pi}{3}\sin\dfrac{4\pi}{9} = \dfrac{3}{16}$ (iii) $\sin 10^0 \sin 30^0 \sin 50^0 \sin 70^0 = \dfrac{1}{16}$

REVIEW EXERCISE 10

1. **Choose the correct option**

(i) $\cos 50^0\, 50' \cos 9^0\, 10' - \sin 50^0\, 50' \sin 9^0\, 10' =$

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{\sqrt{3}}{2}$

(ii) If $\tan 15^0 = 2-\sqrt{3}$, then the value of $\cot^2 75^0$ is

(a) $7+\sqrt{3}$ (b) $7-2\sqrt{3}$ (c) $7-4\sqrt{3}$ (d) $7+4\sqrt{3}$

(iii) If $\tan(\alpha+\beta) = 1/2$ and $\tan\alpha = 1/3$, then $\tan\beta =$

(a) 1/6 (b) 1/7 (c) 1 (d) 7/6

(iv) $\sin\theta\cos(90^0-\theta) + \cos\theta\sin(90^0-\theta) =$ ____

(a) –1 (b) 2 (c) 0 (d) 1

(v) Simplified expression of $(\sec\theta+\tan\theta)(1-\sin\theta)$ is

(a) $\sin^2\theta$ (b) $\cos^2\theta$ (c) $\tan^2\theta$ (d) $\cos\theta$

(vi) $\sin\left(x-\dfrac{\pi}{2}\right) = ?$

(a) $\sin x$ (b) $-\sin x$ (c) $\cos x$ (d) $-\cos x$

(vii) A point is in Quadrant-III and on the unit circle. If its x-coordinate is $-\frac{4}{5}$, what is the y-coordinate of the point?

(a) 3 / 5 (b) –3 / 5 (c) –2 / 5 (d) 5 / 3

(viii) Which of the following is an identity?

(a) sin (a) cos (a) = (1/2) sin(2a) (b) sina + cosa = 1

(c) sin(–a) = sin a (d) tana = cosa / sina

Prove the following identities:

2. $\dfrac{2\sin\theta\sin 2\theta}{\cos\theta+\cos 3\theta} = \tan 2\theta\tan\theta$ 3. $\dfrac{\sin 10a-\sin 4a}{\sin 4a+\sin 2a} = \dfrac{\cos 7a}{\cos a}$

4. $\sin^2\frac{\theta}{2} = \frac{\sin\theta\tan\frac{\theta}{2}}{2}$

5. $\tan\theta\cdot\tan\frac{\theta}{2} = \sec\theta - 1$

6. $\cos 4\theta = 1 - 8\sin^2\theta\cos^2\theta$

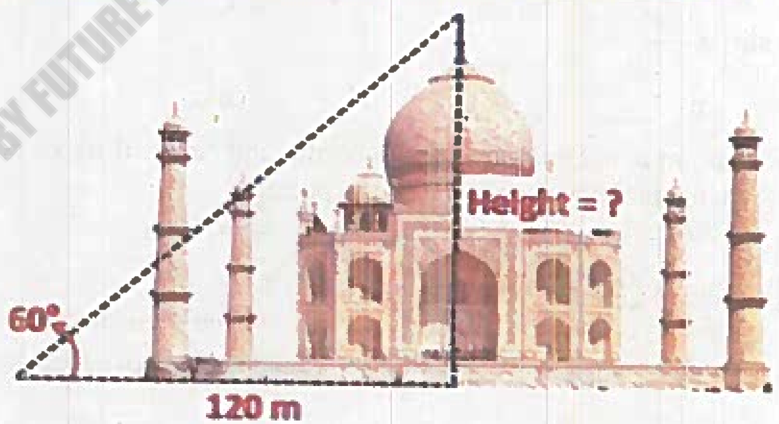
7. $\sin 6x\sin x + \cos 4x\cos 3x = \cos 3x\cos 2x$

8. Prove that $\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right) = \frac{1}{2}\cos 2\theta$

9. Prove that i) $\frac{\sin^2(\pi+\theta)\tan\left(\frac{3\pi}{2}+\theta\right)}{\cot^2\left(\frac{3\pi}{2}-\theta\right)\cos^2(\pi-\theta)\operatorname{cosec}(2\pi-\theta)} = \cos\theta$

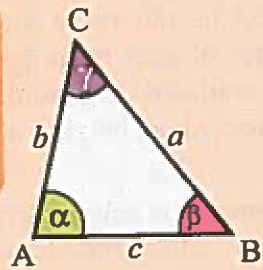
ii) $\frac{\cos(90^0+\theta)\sec(-\theta)\tan(180^0-\theta)}{\sec(360^0-\theta)\sin(180^0+\theta)\cot(90^0-\theta)} = -1$

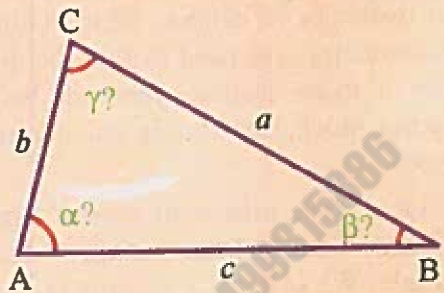
Real Life Applications of Trigonometry



UNIT 11

APPLICATION OF TRIGONOMETRY





STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Solve right angled triangle when measures of
  - two sides are given,
  - one side and one angle are given.
- Define an oblique triangle and prove
  - the law of cosines,
  - the law of sines,
  - the law of tangents, and deduce respective half angle formulae.
- Apply above laws to solve oblique triangles.
- Derive the formulae to find the area of a triangle in terms of the measures of
  - two sides and their included angle,
  - one side and two angles,
  - three sides (Hero's formula)
- Define circum-circle, in-circle and escribed-circle.
- Derive the formulae to find
  - circum-radius,
  - in-radius,
  - escribed-radii, and apply them to deduce different identities.

11.1 Introduction

Trigonometry has an enormous variety of application. It is used extensively in a number of academic fields, primarily mathematics, science and engineering.

Trigonometry, in ancient times, was often used in the measurement of heights and distances of objects which could not be otherwise measured. For example, trigonometry was used to find the distance of stars from the earth. Even today, in spite of more accurate methods being available, trigonometry is often used for making quick and simple calculations regarding heights and distance of far-off objects.

One of the important uses of trigonometry is solving triangles. Every triangle has three sides and three angles, which are called the elements (or parts) of the triangle. We say that a triangle is solved when all six elements are known and listed. Typically three elements, in which one is side, will be given and it will be our task to find the other three elements using trigonometric laws and definitions.

As shown in figure 11.1 we use standard lettering for naming the sides and angles of a right triangle, side a is opposite to angle A, side b is opposite to angle B, where a and b are the legs, and side c, the hypotenuse, is opposite to angle C, the right angle.

A triangle is usually labeled as shown in figure 11.1

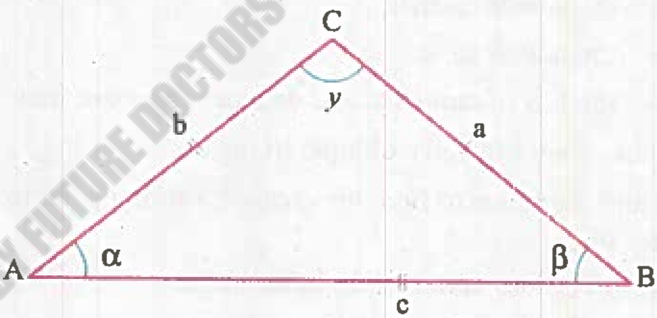


Figure 11.1

The vertices are labeled A, B, C with sides opposite to these vertices are denoted by a,b,c respectively and the measure of three angles are usually denoted by α, β and γ respectively.

We begin with, using the trigonometric functions to solve right angled triangles. Later we will learn how to solve triangles that are not necessarily right angled triangles. We will also derive formulae for finding the areas of such triangles.

11.1.1 Solution of Right Angled Triangles

We can solve a right angled triangle provided that either measure of

(i) two sides are given or (ii) one acute angle and one side are given. We consider the cases as follows:

Case-I: When measure of two sides are given

Example 1: Solve the right angled triangle ABC, in which $a = 15$, $c = 17$ and $\gamma = 90°$.

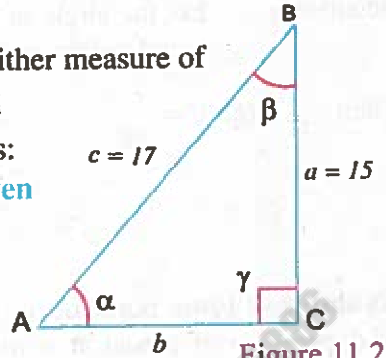


Figure 11.2

Solution: From figure 11.2, we have

$$\sin\alpha = \frac{a}{c} = \frac{15}{17} = 0.882$$

$\Rightarrow$ $\alpha = \sin^{-1}(0.882) = 61.89°$

since $\alpha + \beta = 90°$

$\Rightarrow$ $\beta = 90° - \alpha$

$= 90° - 61.89° = 28.11°$

Now $\cos\alpha = \dfrac{b}{c}$

$\Rightarrow$ $b = c\cos\alpha = 17\cos(61.89°)$

$= 17(0.471)$

$= 8$

Note

The side b can also be found by using Pythagorean Theorem
$c^2 = a^2 + b^2$ or
$b^2 = c^2 - a^2 = \sqrt{(17)^2 - (15)^2}$
so that $b = 8$.

Case-II: When measure of one angle and one side are given

Example 2: Solve the right angled triangle ABC, in which $b = 12$, $\alpha = 70°$ and $\gamma = 90$

Solution: From figure 11.3, we have

$\tan 70° = \dfrac{a}{12}$.

or $a = 12\tan 70°$

$= 12(2.747)$

$= 32.97$ ft.

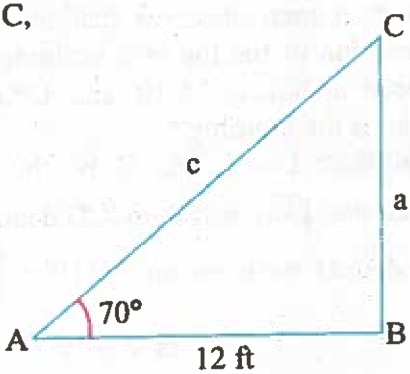


Figure 11.3

To find the length c of the ladder we have

$\cos 70° = \dfrac{12}{c}$

or $c = 12\sec 70°$

$= 12(2.92)$

$= 35.088$ ft.

Example 3: The angle of elevation of a tree from a point on the ground 42m from its base is $33°$. Find the height of the tree?

Solution: Let the angle of elevation = θ
and height of the tree $= h$

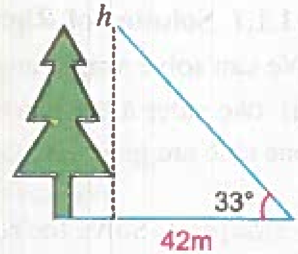


Figure 11.4

Then $\tan\theta = \dfrac{h}{42} \Rightarrow \tan 33° = \dfrac{h}{42}$

$\Rightarrow h = 42\tan 33°$

≈ 27.28

The tree is 27m tall.

Example 4: From point B, the top of a light house 120 ft above the sea, the angle of depression of a boat at point A is 5°. How far is it from the light house to the boat?

Solution: Since the angle of depression is the acute angle formed by the line of sight and the horizontal line passing through the position of sighting. Figure 11.5 indicates the situation. The angle A must also be 5° in measure. We have

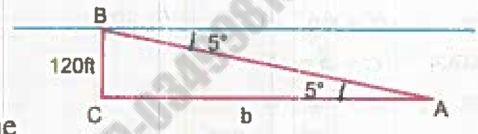


Figure 11.5

$\cot A = \dfrac{b}{120}$ or $\cot 5° = \dfrac{b}{120}$

$\Rightarrow b = 120\ (11.43) = 1372$ ft, approx.

Example 5: From the two successive positions on a straight road 1000 meter apart, a man observes that the angle of elevation of the top of a building directly ahead of him is 12°10′ and 42°35′. How high is the building?

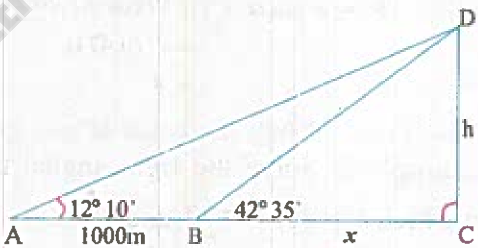


Figure 11.6

Solution: Let A and B be the two successive positions of a man on the road such that $|\overline{AB}| = 1000$m. CD denote the height h of the building and let BC $= x$

In ΔACD we have $\tan 12°10' = \dfrac{CD}{AC} = \dfrac{h}{AB+BC} = \dfrac{h}{x+1000}$

or $x + 1000 = h\cot 12°10' = 4.6382\,h$ (1)

In ΔBCD we have $\tan 42°35' = \dfrac{h}{x}$

$\Rightarrow x = h\cot 42°35' = 1.088\,h$ (2)

From (1),(2) $1.088h + 1000 = 4.6382\,h$

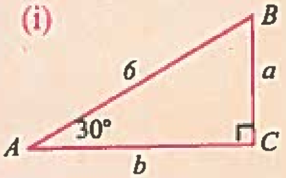
$\Rightarrow h = 281.67\text{m} \simeq 282$ m Now from (2) we get

$x = 306.8 \simeq 307$m

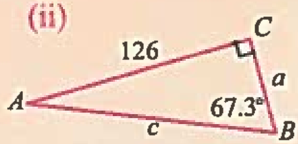
EXERCISE 11.1

1. Solve the following right triangles.

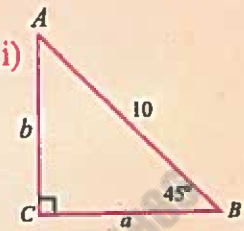
(i)



(ii)



(iii)



2. Solve right triangles ABC in which $\gamma = 90^\circ$ and
   (i) $a = 14$, $\beta = 28^\circ$ (ii) $b = 8.9$, $\beta = 21.5^\circ$ (iii) $b = 14$, $c = 450$

3. The angle of elevation of the top of a post from a point on level ground 38m away is 33.23°. Find the height of the post.

4. A masjid minar 82 meters high casts a shadow 62 meters long. Find the angle of elevation of the sun at that moment.

5. The angle of depression of a boat 65.7m from the base of a cliff is 28.9°. How high is the cliff?

6. From the top of a cliff 52m high the angles of depression of two ships due east of it are 36° and 24° respectively. Find the distance between the ships.

7. Two masts are 20m and 12m high. If the line joining their tops makes an angle of 35° with the horizontal; find their distance apart.

8. The measure of the angle of elevation of a kite is 35°. The string of the kite is 340 meters long. If the sag in the string is 10 meters, find the height of the kite.

9. A parachutist is descending vertically. How far does the parachutist fall as the angle of elevation changes from 50° to 30° which observes from a point 100m away from the feet of parachutist where he touches the ground.

10. An isosceles triangle has a vertical angle of 108° and a base 20 cm long. Calculate its altitude.

11.1.2 Oblique Triangles

If none of the angle of a triangle is right angle, the triangle is called oblique triangle. In Figure 11.7 both triangles are oblique triangles.

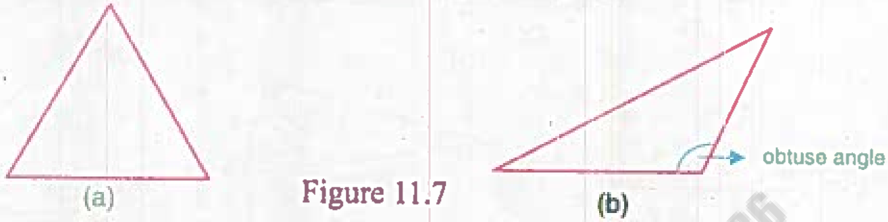


Figure 11.7

We see that an oblique triangle has either

(i) three acute angles (figure 11.7(a)) or

(ii) two acute angles and one obtuse angle (figure 11.7(b))

In the last section we solved right angled triangles, however, in this section we will solve oblique triangles. Given three elements of a triangle we will be asked to find the remaining three elements. Thus, we have the following five possibilities:

When three parts of a triangle including at least one side are known, the triangle is uniquely determined. The five cases of oblique triangles are

1. A.A.S: Given two angles and the side opposite to one of them
2. A.S.A: Given two angles and the included side
3. S.S.A: Given two sides and the angle opposite to one of them
4. S.A.S: Given two sides and the included angle
5. S.S.S: Given the three sides

In case of (S.S.A) there is not always a unique solution. It is possible to have no solution for the angle, one solution for the angle, or two solutions——an angle and its supplement.

In order to solve the above cases of oblique triangles, we develop special mathematical tools called the law of cosines, the law of sines the law of tangents.

(a) The Law of Cosines

In this section, we will derive the law of cosines and we use it to solve the case 5 of oblique triangles.

Theorem (Law of Cosines) In any triangle with usual labelling

(i) $a^2 = b^2 + c^2 - 2bc\cos\alpha$

(ii) $b^2 = c^2 + a^2 - 2ca\cos\beta$

(iii) $c^2 = a^2 + b^2 - 2ab\cos\gamma$

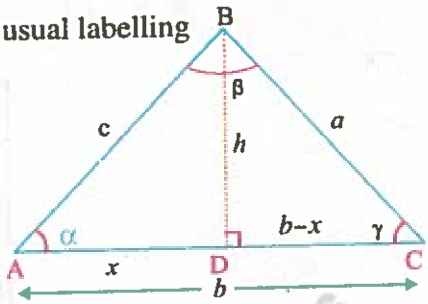


Figure 11.8

Proof: Case 1: All the angles are acute, α is an acute angle in figure 11.8. If h is the altitude of vertex B, then in ΔBCD, we have,

$$a^2 = h^2 + (b-x)^2 \qquad (1)$$

In ΔBAD we have

$$\cos\alpha = \frac{x}{c}$$

$$\therefore x = c\cos\alpha \qquad (2)$$

and $\quad c^2 = x^2 + h^2 \qquad (3)$

Put (2) and (3) in (1)

$$a^2 = (c^2 - x^2) + (b^2 - 2bx + x^2)$$

$$= b^2 + c^2 - 2bc\cos\alpha$$

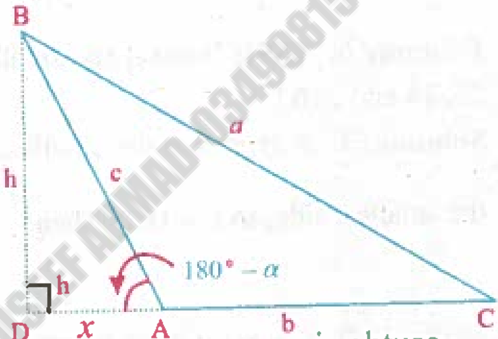


Figure 11.9

Case2: One angle is obtuse. α is obtuse here

In ΔBCD $\quad a^2 = h^2 + (b+x)^2$

giving $\quad a^2 = h^2 + b^2 + x^2 + 2bx \qquad (1)$

In ΔBAD, $\quad \cos(180^\circ - \alpha) = \frac{x}{c}$

$$\therefore x = c\cos(180^\circ - \alpha) = -c\cos\alpha \qquad (2)$$

and $\quad c^2 = h^2 + x^2 \qquad (3)$

Put (2) and (3) into (1)

$$a^2 = (c^2 - x^2) + b^2 + x^2 + 2b(x) = b^2 + c^2 + 2b(-c\cos\alpha) = b^2 + c^2 - 2bc\cos\alpha$$

In both the triangles, we obtained $a^2 = b^2 + c^2 - 2bc\cos\alpha$.

By considering angles B and C in a similar manner, we can prove that

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

By rearranging the formula we can express the cosine of the angles in terms of three lengths sides of the triangle.

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$
$$\cos\beta = \frac{a^2+c^2-b^2}{2ac}$$
$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab}$$

Did You Know ?

S.S.S. and S.A.S. possibilities could be tackled by using cosine law. However in S.A.S., where two sides and included angle is given, it is necessary that the given angle must be less than 180°.

Example 6: (SSS): What is the smallest angle of a triangle whose sides measure 25, 18 and 21ft?

Solution: If γ represent the smallest angle, then c (the side opposite) γ must be the smallest side, so c = 18. Then $\cos\gamma = \frac{a^2+b^2-c^2}{2ab} = \frac{(25)^2+(21)^2-(18)^2}{2(25)(21)} = 0.707$

$$\Rightarrow \gamma = \cos^{-1}(0.707) = 45°$$

Example 7: (S.A.S.): Find c where $a = 52$, b = 28.3, $\gamma = 38.5°$

Solution: γ is the angle included between a and b.

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$
$$= (52)^2 + (28.3)^2 - 2(52)(28.3)\cos 38.5°$$
$$\Rightarrow c^2 \simeq 918.355$$
$$\Rightarrow c \simeq 30.30 \text{ unit}$$

Example 8: A body is acted upon by the forces 10N and 20N making an angle $25°35'$ with each other. Find the magnitude of the resultant of the forces.

Solution:

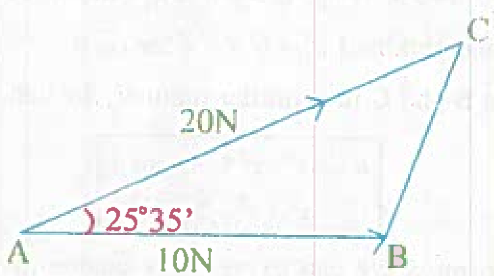


Figure 11.10 (a)

The forces of 10N and 20N are represented by sides of parallelogram.

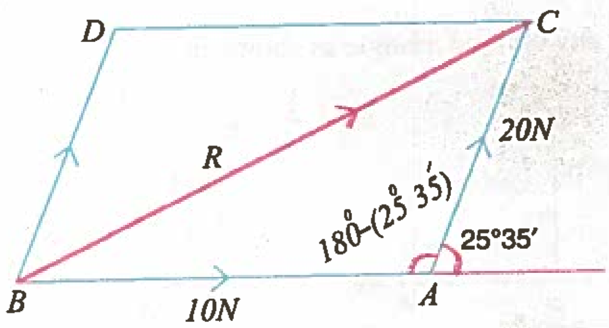


Figure 11.10 (b)

The resultant R is the diagonal of parallelogram ABCD. Hence

$$R^2 = (10)^2 + (20)^2 - 2\times10\times20\cos(180° - 25°35')$$

$$= 860.78\,N^2 \Rightarrow R = 29.3 \text{ N}.$$

Example 9: An equilateral triangle is inscribed in a circle of radius 5cm. Find the perimeter of the triangle.

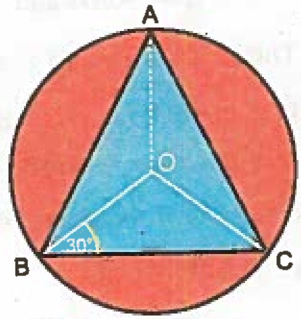


Figure 11.11

Solution: Let O be the centre of the circle. Join O with vertices B and C.

In the equilateral triangle ABC, we have

$$\angle BOC = \angle AOC = \angle AOB = \frac{1}{3}(360°) = 120°$$

$$|\overline{OB}| = |\overline{OC}| = |\overline{OA}| = 5cm$$

Using cosine law

$$|\overline{BC}|^2 = |\overline{OB}|^2 + |\overline{OC}|^2 - 2\times|\overline{OB}||\overline{OC}|\cos\angle BOC$$

$$= 5^2 + 5^2 - 2\times5\times5\cos120°$$

$= \sqrt{75}$ .Each side is $\sqrt{75}$ cm.

Hence perimeter of $\Delta ABC = \sqrt{75} + \sqrt{75} + \sqrt{75} = 3\sqrt{75} = 15\sqrt{3}cm$

(b) The Law of sines

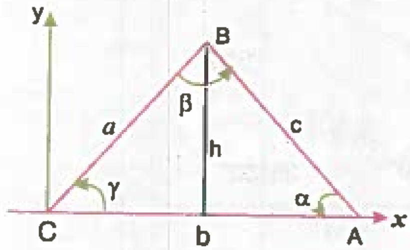
In the last section we discussed the two possibilities of solving oblique triangles SSS, SAS.

In this section we will consider the fourth case ASA or AAS which is one case because knowing any two angles and one side means knowing all the three angles and one side. The law of cosine does not work where at least two sides are needed. We state and prove the law of sines for this purpose.

Theorem: In any Δ ABC with usual labelling

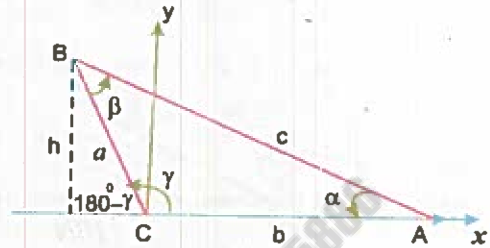
$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$$

Proof: Consider any oblique triangle as shown in figure 11.12



(i) γ is acute

Figure 11.12(a)



(ii) γ is obtuse

Figure 11.12(b)

Let h = height of the triangle with base $\overline{CA}$. Then in figure 11.12 (a)

$$\sin\alpha=\frac{h}{c} \text{ and } \sin\gamma=\frac{h}{a} \qquad \text{(Solving for h)}$$

$$h=c\sin\alpha \text{ and } h=a\sin\gamma$$

Thus $c\sin\alpha=a\sin\gamma \Rightarrow \dfrac{\sin\alpha}{a}=\dfrac{\sin\gamma}{c}$ (1)

In figure 11.12 (b) $h=a\sin(180^\circ-\gamma)=a\sin\gamma$ and $h=c\sin\alpha$

Hence $c\sin\alpha=a\sin\gamma$

Similarly if we draw perpendiculars from the other two vertices on opposite sides of ΔABC we get

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b} \qquad (2)$$

and

$$\frac{\sin\beta}{b}=\frac{\sin\gamma}{c} \qquad (3)$$

Combining (1), (2) and (3) we have

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$$

or equivalently,

$$\boxed{\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}}$$

These equations give **the law of sines.**

Example 10: For a triangle ABC, given $a = 30$, $b = 70$ $\beta = 85°$. Find α.

Solution: Using law of sine

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

$$\Rightarrow \qquad \sin\alpha = 30 \times \frac{\sin 85°}{70} = 0.4269 \Rightarrow \alpha = 25°16'25''$$

Example 11: From a point A the angle of elevation of the top C of a tower is 28°. From a second point B, which is 2200 ft closer to the base of the tower, the angle of elevation of the top is 66°. What is the height h of the tower?

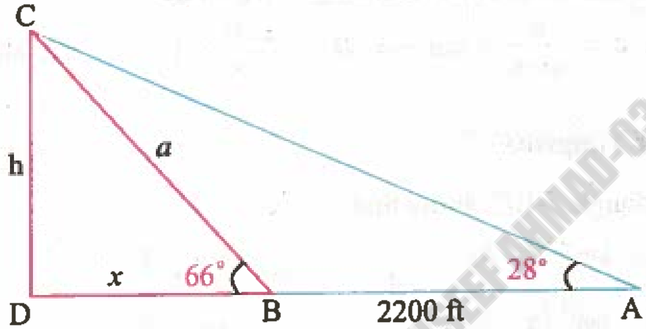


Figure 11.13

Solution: For ΔABC, $AB = 2200$, $\angle ABC = 180° - 66° = 114°$ and $\angle BCA = 38°$.
Applying the law of sines to ΔABC, we have

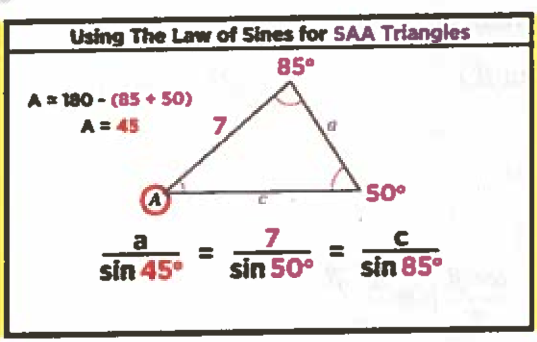
$$\frac{\sin 38°}{2200} = \frac{\sin 28°}{a}$$

Thus $a = 1678$ ft, approximately.

Now for ΔBDC, we have

$$\sin 66° = \frac{h}{a} = \frac{h}{1678}$$

Thus $h = 1678 \sin 66° = 1533$ ft, approximately.



Example 12: Solve the triangle in which

$\alpha = 38^\circ, \beta = 121^\circ$ *and* $a = 20$

Solution: Since $\alpha + \beta + \gamma = 180^\circ$

$\gamma = 180^\circ - 121^\circ - 38^\circ = 21^\circ$ Use the law of sines to get b

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha}$$

$$b = \frac{a \cdot \sin\beta}{\sin\alpha} \Rightarrow b = 20 \times \frac{\sin 121^\circ}{\sin 38^\circ} = 28 \text{ approximately.}$$

Use the law of sines again but this time using α , γ to get

$$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha} \Rightarrow c = \frac{a}{\sin\alpha} \times \sin\gamma = 20 \times \frac{\sin 21^\circ}{\sin 38^\circ} = 11.6 \approx 12 \text{ approximately.}$$

(c) The Law of Tangents

Theorem: In any triangle ABC, show that

(i) $$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}$$ (ii) $$\frac{b+c}{b-c} = \frac{\tan\frac{1}{2}(\beta+\gamma)}{\tan\frac{1}{2}(\beta-\gamma)}$$

(iii) $$\frac{c+a}{c-a} = \frac{\tan\frac{1}{2}(\gamma+\alpha)}{\tan\frac{1}{2}(\gamma-\alpha)}$$

Proof: By the law of sines in any triangle ABC

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = D \text{ (say)}$$

We have

$a = D\sin\alpha$ and $b = D\sin\beta$

$a + b = D(\sin\alpha + \sin\beta)$ (1)

$a - b = D(\sin\alpha - \sin\beta)$ (2)

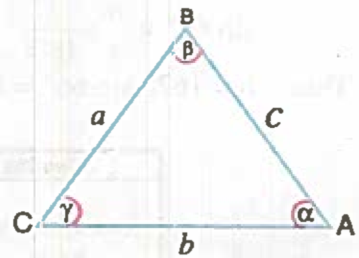


Figure 11.14

From (1) and (2)

$$\frac{a+b}{a-b} = \frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta}$$

Using the formulae

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

and $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

we get

$$\frac{a+b}{a-b} = \frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}} \Rightarrow \boxed{\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}}$$

Similarly $\boxed{\frac{b+c}{b-c} = \frac{\tan\frac{1}{2}(\beta+\gamma)}{\tan\frac{1}{2}(\beta-\gamma)}}$ and $\boxed{\frac{c+a}{c-a} = \frac{\tan\frac{1}{2}(\gamma+\alpha)}{\tan\frac{1}{2}(\gamma-\alpha)}}$

These three relations are known as the law of tangents. Note that the interchange of lengths a, b result in the interchange of angles α, β. Hence if b>a then it is better to use the tangent formula in the form.

$$\frac{b+a}{b-a} = \frac{\tan\frac{1}{2}(\beta+\alpha)}{\tan\frac{1}{2}(\beta-\alpha)}$$

Example 13: Use the law of tangents to solve the triangle ABC in which $a = 925$, $c = 432$ and $\beta = 42°30'$.

Solution: $$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)}$$

But $\alpha + \gamma = 180° - \beta = 137°30' \Rightarrow \frac{1}{2}(\alpha+\gamma) = 68°45'$

Hence $$\frac{925-432}{925+432} = \frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan(68°45')}$$

Therefore $\tan\frac{1}{2}(\alpha-\gamma) = \frac{493}{1357} \times 2.5715 = 0.93 \Rightarrow \frac{1}{2}(\alpha-\gamma) = 43°3' \Rightarrow \alpha-\gamma = 86°6'$

Now $\alpha + \gamma = 137°30'$

$\alpha - \gamma = 86°6'$

By addition $2\alpha = 223°36'$

$\alpha = 111°48'$

By subtraction $2\gamma = 51°24'$

$\gamma = 25°42'$

To find b, we use the law of sines

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$b = \frac{c}{\sin\gamma}\sin\beta = \frac{432}{\sin 25°42'}\sin 42°30' = 673$$

(d) Half Angle Formulae

The half angle formulae are very useful to solve a triangle when the measures of three sides of a triangle are given and no angle is known. These formulae could be derived using the law of cosines.

(i) The cosine of Half the Angle in Terms of Sides

Theorem: In any triangle ABC, show that

$$\cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}} \qquad \cos\frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}$$

$$\cos\frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}} \text{ where } S = \frac{1}{2}(a+b+c)$$

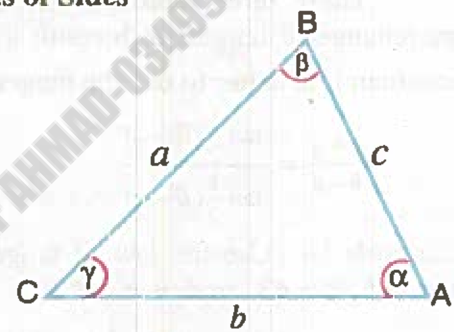


Figure 11.15

Proof: Let $S = \frac{1}{2}(a+b+c)$

Using the law of cosines

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

But $\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1$

Hence $2\cos^2\frac{\alpha}{2} - 1 = \frac{b^2+c^2-a^2}{2bc}$

$$\Rightarrow \quad 2\cos^2\frac{\alpha}{2} = \frac{b^2+c^2-a^2}{2bc} + 1 = \frac{(b+c)^2-a^2}{2bc}$$

The numerator being difference of two squares, can be written as

$$2\cos^2\frac{\alpha}{2} = \frac{[(b+c)+a][(b+c)-a]}{2bc}$$

Since $a+b+c = 2S$

and $b+c-a = 2S - 2a = 2(S-a)$.

Hence $2\cos^2\frac{\alpha}{2} = \frac{(2S)\times 2(S-a)}{2bc}$

$$\Rightarrow \quad \cos^2\frac{\alpha}{2} = \frac{S(S-a)}{bc} \text{ or } \cos\frac{\alpha}{2} = \pm\sqrt{\frac{S(S-a)}{bc}}$$

As $\alpha \angle 180°$, $\frac{\alpha}{2}$ is a measure of acute angle, the value of $\cos\frac{\alpha}{2}$ will be positive.

Hence $\boxed{\cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}}$ (1)

Similarly we can prove

$$\boxed{\cos\frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}}}$$

$$\boxed{\cos\frac{\gamma}{2} = \sqrt{\frac{S(S-c)}{ab}}}$$

(ii) The sines of Half the Angle in Terms of Sides

Theorem: In any triangle ABC, show that

$$\sin\frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}$$

$$\sin\frac{\beta}{2} = \sqrt{\frac{(S-c)(S-a)}{ac}}$$

$$\sin\frac{\gamma}{2} = \sqrt{\frac{(S-a)(S-b)}{ab}}, \text{ where } S = \frac{1}{2}(a+b+c)$$

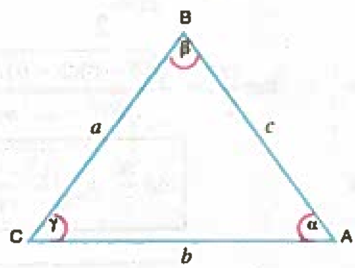


Figure 11.16

Proof: $\cos\alpha = 1-2\sin^2\frac{\alpha}{2}$

Hence $2\sin^2\frac{\alpha}{2} = 1-\cos\alpha$

$$=1-\frac{b^2+c^2-a^2}{2bc} = \frac{a^2-(b-c)^2}{2bc}$$

$$= \frac{(a-b+c)(a+b-c)}{2bc}$$

Since $a+b+c = 2S$

so $a-b+c = 2S-2b = 2(S-b)$ and $a+b-c = 2S-2c = 2(S-c)$.

Substituting these values in the above equation

$$2\sin^2\frac{\alpha}{2} = \frac{2(S-b)\times 2(S-c)}{2bc} \quad \Rightarrow \sin\frac{\alpha}{2} = \pm\sqrt{\frac{(S-b)(S-c)}{bc}}$$

Again $\sin\frac{\alpha}{2}$ is measure of an acute angle $\sin\frac{\alpha}{2}$ is always positive.

Hence $\boxed{\sin\frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}}$ (2)

Similarly $\sin\frac{\beta}{2} = \sqrt{\frac{(S-c)(S-a)}{ac}}$ and $\sin\frac{\gamma}{2} = \sqrt{\frac{(S-a)(S-b)}{ab}}$

(iii) The Tangent of Half the Angle in Terms of the Sides

Theorem: In any triangle ABC, show that

$$\tan\frac{\alpha}{2} = S\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$$

$$\tan\frac{\beta}{2} = S\sqrt{\frac{(S-c)(S-a)}{S(S-b)}}$$

$$\tan\frac{\gamma}{2} = S\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$$

where $S = \frac{1}{2}(a+b+c)$

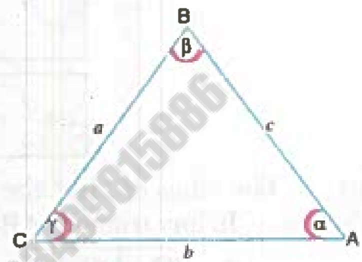


Figure 11.17

Proof: $\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$

$\Rightarrow \quad \tan\frac{\alpha}{2} = \frac{\sqrt{(S-c)(S-b)/bc}}{\sqrt{S(S-a)/bc}}$ (by (1) and (2))

$\Rightarrow \quad \tan\frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$ (3)

Similarly $\tan\frac{\beta}{2} = \sqrt{\frac{(S-c)(S-a)}{S(S-b)}}$

$\tan\frac{\gamma}{2} = \sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$

Now if we multiply and divide the right hand side of (3) by $\sqrt{(S-a)}$ we get

$$\tan\frac{\alpha}{2} = \frac{1}{(S-a)}\sqrt{\frac{(S-a)(S-b)(S-c)}{S}}$$

Denoting $\sqrt{\frac{(S-a)(S-b)(S-c)}{S}}$ by r, we get

$\tan\frac{\alpha}{2} = \frac{r}{S-a}$ (4)

Similarly $\tan\frac{\beta}{2} = \frac{r}{S-b}$ and $\tan\frac{\gamma}{2} = \frac{r}{S-a}$

where $S = \frac{1}{2}(a+b+c)$ and $r = \sqrt{\frac{(S-a)(S-b)(S-c)}{S}}$

Example 14: Solve the triangle ABC with usual notation for its sides given that $a = 75$, $b = 55$ and $c = 50$

Solution: $S = \frac{1}{2}(a+b+c) = \frac{1}{2}(75+55+50)$

$= 90$

So $S - a = 90 - 75 = 15$

$S - b = 90 - 55 = 35$

$S - c = 90 - 50 = 40$

Using half angle formula

$$\cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}} = \sqrt{\frac{90(15)}{(55)(50)}} = 0.700649$$

$\Rightarrow \frac{\alpha}{2} = 45°31'$ or $\alpha = 91°2'$

Also $\cos\frac{\beta}{2} = \sqrt{\frac{S(S-b)}{ac}} = \sqrt{\frac{90\times35}{75\times50}} = 0.9165$

$\Rightarrow \frac{\beta}{2} = 23°35'$, $\beta = 47°10'$

Hence $\gamma = 180° - (\alpha + \beta) = 180° - 138°12' = 41°48'$

EXERCISE 11.2

1. Solve the triangles with dimensions.

(i) $a = 209$, $b = 120$, $c = 241$ (ii) $a = 120$, $b = 240$, $\gamma = 32°$

(iii) $\alpha = 100°$, $c = 345$, $\gamma = 56.4°$ (iv) $a = 24.5$, $c = 43.8$, $\beta = 112°$

(v) $b = 1.6$, $c = 3.2$, $\alpha = 100°24'$ (vi) $\beta = 39°30'$, $\gamma = 34°10'$, $a = 240$

(vii) $\alpha = 35°$, $\beta = 70°$, $c = 115$ (viii) $a = 37.5$, $b = 12.4$, $\beta = 72°$

(ix) $b = 12.5$, $c = 23$, $\alpha = 38°20'$ (x) $a = 168$, $c = 319$, $\beta = 110°22'$

2. Find the angle of largest measure (Using half sine law).

(i) $a = 74$, $b = 52$ and $c = 47$

(ii) $a = 7$, $b = 9$ and $c = 7$

(iii) $a = 2.3$, $b = 1.5$ and $c = 2.7$

3. Solve the triangle for which length of three sides are given.(Using half cosine law)

   (i) $a = 9$, $b = 7$ and $c = 5$

   (ii) $a = 1.2$, $b = 9$ and $c = 10$

   (iii) $a = 6$, $b = 8$ and $c = 12$

4. One diagonal of a parallelogram is 20cm long and at one end forms angles 20° and 40° with the sides of the parallelogram. Find the length of the sides.

5. Two planes start from Karachi International Airport at the same time and fly in directions that make an angle of 127° with each other. Their speeds are 525km/h. How far apart they are at the end of 2 hours of flying time?

6. A city block is bounded by three streets. If the measure of the sides of the block are 285,375 and 396 meters, find the measure of the angles of the street make with each other.

7. The diagonal of a parallelogram meets the sides at angle of 30° and 40°. If the length of the diagonal is 30.0cm, then find the perimeter of the parallelogram.

8. Use the law of cosines to prove

   (i) $$1+\cos\alpha = \frac{(b+c+a)(b+c-a)}{2bc}$$

   (ii) $$1-\cos\alpha = \frac{(a-b+c)(a+b-c)}{2bc}$$

11.2 Areas of Triangular Regions

To find the area of a triangle ABC we discuss three cases SAS, SAA and SSS separately as follow

(a) Area of a triangle when two sides and their included angle is given.

From elementary geometry we know that the area of a triangle is equal to

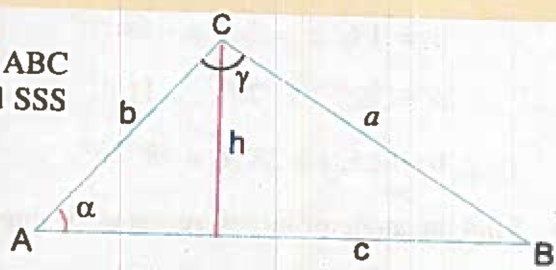


Figure 11.18

one half the product of measure of the base and measure of altitude. In figure 11.18 for the triangle ABC. Let h be the measure of altitude.

Then area Δ is given by $\Delta = \frac{1}{2}$ (AB)(h)

But since AB = c and $\frac{h}{b} = \sin\alpha$ or $h = b\sin\alpha$

$$\therefore \quad \Delta = \frac{1}{2}c(b\sin\alpha) = \frac{1}{2}bc\sin\alpha \qquad (1)$$

Also h can be written as $\frac{h}{a} = \sin\beta$ or $h = a\sin\beta$

So that Δ becomes, $\Delta = \frac{1}{2}(c)(a\sin\beta) = \frac{1}{2}ac\sin\beta$

Similarly by taking other sides of the triangle ABC as base

We have $\Delta = \frac{1}{2}ab\sin\gamma$

Hence the area Δ can be found by either formula

$$\Delta = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ac\sin\beta = \frac{1}{2}bc\sin\alpha$$

This shows that the area of a triangle is

"One half the product of the measure of two sides and the sine of the measure of the angle included between them."

(b) Area of a triangle when the measure of one side and measure of two angles is given (SAA).

If in the formula $\frac{1}{2}ac\sin\beta$ of the area of a triangle one of the sides say c is not known we can replace it from the law of sines.

We have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow \quad c = \frac{a\sin\gamma}{\sin\alpha}$$

So that the area is now given by

$$\Delta = \frac{1}{2}ac\sin\beta = \frac{1}{2}a\left(\frac{a\sin\gamma}{\sin\alpha}\right)\times\sin\beta$$

$$\Delta = \frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha} \qquad (2)$$

Similarly we have $\Delta = \frac{1}{2}b^2\frac{\sin\alpha\sin\gamma}{\sin\beta} = \frac{1}{2}\frac{c^2\sin\alpha\sin\beta}{\sin\gamma}$

(c) Area of a triangle when measures of all the sides of a triangle are given.

We know that the area Δ is given by

$$\Delta = \frac{1}{2}\text{bc}\sin\alpha = \frac{1}{2}\text{bc}\times 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \text{bc}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

Using half angle formulae

$$\Delta = \text{bc}\sqrt{\frac{(S-b)(S-c)}{bc}}\times\sqrt{\frac{S(S-a)}{bc}} = \sqrt{S(S-a)(S-b)(S-c)} \qquad (3)$$

This formula is known as **Hero's formula** (alternatively known as Heron's formula).

We now find the area of a triangle by using the above mentioned formula.

Example 15: Find the area of the Δ ABC where $\alpha = 18.4°$, b = 154ft and c = 211ft.

Solution: $\Delta = \frac{1}{2}\text{bc}\sin\alpha = \frac{1}{2}(154)(211)(\sin 18.4°) = 5128.349$

To two decimal places the area is 5128.35 square feet.

Example 16: Find the area of a triangle with angles 20°, 50° and 110° if the side opposite the 50° angle is 24 inches long.

Solution: Let $\alpha = 20°$, $\beta = 50°$ $\gamma = 110°$

Now b is given which is 24 inches

Hence the area Δ is

$$\Delta = \frac{1}{2}b^2\frac{\sin\alpha\sin\gamma}{\sin\beta}$$

$$= \frac{1}{2}(24)^2\frac{\sin 20°\sin 110°}{\sin 50°} = 120.83 \text{ square inches.}$$

Example 17: Find the area of a triangle having sides of 43ft, 89ft and 120ft.

Solution: Since three sides (but none of the angles) are known, we need Hero's formula to find area.

Let $a = 43$, b = 89 and c = 120, then

$$S = \frac{1}{2}(43+89+120) = 126$$

$$\Delta = \sqrt{126(126-43)(126-89)(126-120)} \simeq 1523.70$$

To two decimal places the area is 1523 square ft.

Example 18: What is the vertex angel of an isosceles triangle whose equal sides are 13ft long if the area is 50 ft^2.

Solution: Area Δ ABC $= \frac{1}{2} ab \sin c$

$$50 = \frac{1}{2}(13)(13)(\sin c)$$

$$\sin c = \frac{100}{169} = 0.5917$$

$$c = \sin^{-1}(0.5917) = 36.3°$$
$$= 36°18'$$

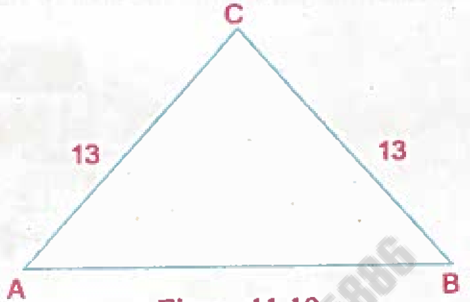


Figure 11.19

EXERCISE 11.3

1. Find the area of the triangle ABC in each case

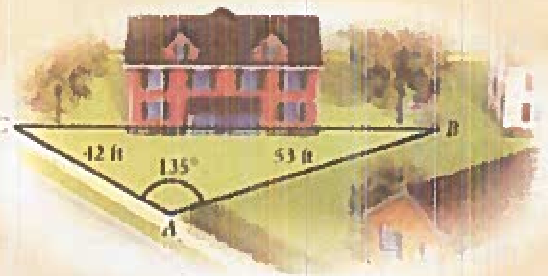
(i)	$a = 15$	$b = 80$	$\gamma = 38°$
(ii)	$b = 14$	$c = 12$	$\alpha = 82°$
(iii)	$a = 30$	$\beta = 50°$	$\gamma = 100°$
(iv)	$b = 40$	$\alpha = 50°$	$\gamma = 60°$
(v)	$a = 7.0$	$b = 8.0$	$c = 2.0$
(vi)	$a = 11$	$b = 9.0$	$c = 8.0$
(vii)	$b = 414$	$c = 485$	$\alpha = 49°47'$
(viii)	$a = 32$	$\beta = 47°24'$	$\gamma = 70°16'$
(ix)	$b = 47$	$\alpha = 60°25'$	$\gamma = 41°35'$
(x)	$c = 57$	$\alpha = 23°24'$	$\beta = 71°36'$
(xi)	$a = 925$	$c = 433$	$\beta = 42°17'$
(xii)	$a = 92$	$b = 71$	$\gamma = 56°44'$

2. The area of triangle is 121.34. If $\alpha = 32°25'$, $\beta = 65°65'$ then find c and angle γ.

3. One side of a triangular garden is 30 m. If its two corner angles are $22\frac{1}{2}$ and $112\frac{1}{2}$, find the cost of planting the grass at the rate of Rs. 5 per square meter.

4. A new home owner has a triangular-shaped back yard. Two of the three sides measure 53 ft and 42 ft and form an included angle of 135°. To determine the

amount of fertilizer and grass seed to be purchased, the owner has to know the area of the yard. Find the area of the yard to the nearest square foot.



11.3 Circles Connected with Triangles

11.3.1 (a) **Circumcircle:** A circle passing through the vertices of any triangle is called the circumcircle. The measure of radius of this circle called **circumradius** and is denoted by R. The center of this circle is called **circumcenter**.

The circumcenter is the point where the right bisectors of its sides meet each other.

(b) **Incircle:** A circle drawn inside a triangle and touching its sides is called the incircle associated with the triangle. Its radius is called **inradius** and its center is called **incenter**.

The student knows from elementary geometry that incenter is the point at which internal bisectors of the angles of a triangle meet each other.

(c) **Escribed Circles:** A circle, which touches one side of a triangle externally, and the other two sides internally when produced is called **escribed circle** or **ex-circle** or **e-circle**.

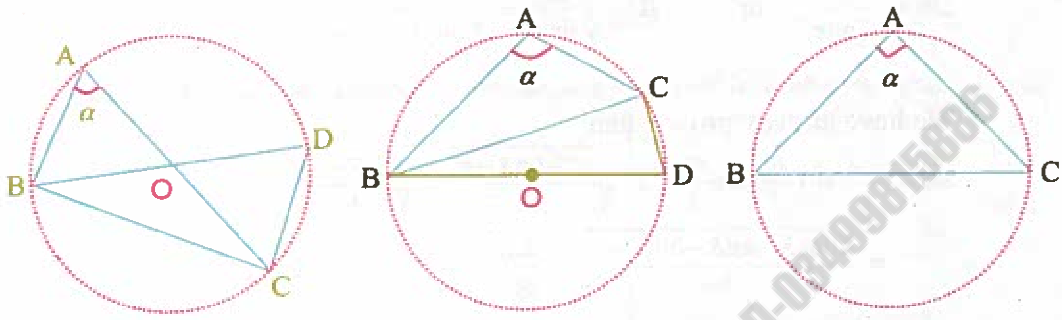
There are three such circles, touching the sides a, b and c externally. Each circle is associated with the side of the triangle it touches externally. The measure of the radius of the circle opposite to the vertex (touching side externally) is denoted by r_1 and measures of the radius of the circles opposite to the vertices B and C are denoted by r_2 and r_3 respectively. The centres of these circles called **ex-centres** are similarly denoted by I_1, I_2 and I_3.

The ex-centre I_1 with respect to the vertex A is the point of intersection of the external bisectors of angles B and C and internal bisector of angle A.

11.3.2 (a)To find circumradius for any triangle ABC

(i) To find R, the circumradius of a triangle ABC in terms of measure of a side and its opposite angle.

Let O be the circumcenter of the triangle ABC. Join B and O and produce it to meet the circle at D. Join C and D.



(i) (ii) (iii)

Figure 11.20

Figure 11.20 (i), (ii) and (iii) depicts the cases where measure of angle α is acute, obtuse and right angle respectively.

Now measure of $\overline{BD}$ is the diameter of circumcircle. Hence

$$\overline{BD} = 2R$$

$$m\overline{BC} = a$$

In figure (i) m $\angle BDC = \alpha < \frac{\pi}{2}$

Because α and $\angle BDC$ are angels in the same area of circle made by chord $\overline{BC}$.

Hence $\frac{m\overline{BC}}{m\overline{BD}} = \sin\angle BDC = \sin\alpha$

So $\frac{a}{2R} = \sin\alpha$ (1)

In figure (ii) $\angle BDC$ and $\angle\alpha$ are supplementary angles because they are made by the same chord $\overline{BC}$ in two opposite arcs $\overset{\frown}{BAC}$ and $\overset{\frown}{BDC}$.

Hence

$$\frac{m\overline{BC}}{m\overline{BD}} = \sin\angle BDC = \sin(\pi - \alpha)$$

$$= \sin\alpha \quad (2)$$

In figure (iii) $\alpha = \frac{\pi}{2}$

Hence in this case

$$\frac{m\overline{BC}}{m\overline{BD}} = 1 = \sin\frac{\pi}{2} = \sin\alpha \text{ i.e. } \frac{R}{2a} = \sin\alpha \qquad (3)$$

Hence all the three situations lead to the conclusion that

$$2R = \frac{a}{\sin\alpha} \quad \text{or} \quad R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$$

(ii) Circumradius in terms of the measurements of sides of a triangle

We have already proved that

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 2\sqrt{\frac{(S-b)(S-c)}{bc}} \times \sqrt{\frac{S(S-a)}{bc}}$$

$$= \frac{2\sqrt{S(S-a)(S-b)(S-c)}}{bc} = \frac{2\Delta}{bc}$$

$$R = \frac{a}{2\sin\alpha} = \frac{abc}{4\Delta}$$

where $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$

Example 19: Find the circumscribing radius for a triangle whose sides are 3,5 and 6.

Solution:

$$S = \frac{a+b+c}{2} = \frac{3+5+6}{2} = 7$$

$$R = \frac{abc}{4\sqrt{S(S-a)(S-b)(S-c)}} = \frac{3\times5\times6}{4\sqrt{7(4)(2)(1)}} = \frac{90}{4\sqrt{56}} = \frac{45}{2\sqrt{56}} = 3 \text{ (approx.)}$$

(b) To find inradius r for any triangle ABC

We shall prove $r = \sqrt{\frac{(S-a)(S-b)(S-c)}{S}}$

where $S = \frac{1}{2}(a+b+c)$ is the half perimeter.

Let the internal bisectors of a triangle ABC meet at the point O which is the incenter. Join O with vertices A, B and C. We obtain three triangles OAB, OBC and OCA. The altitude OF, OD and OE respectively of these triangles is a radius of the inscribed circle. The bases of these triangles are sides of the original triangle. Then from figure 11.21

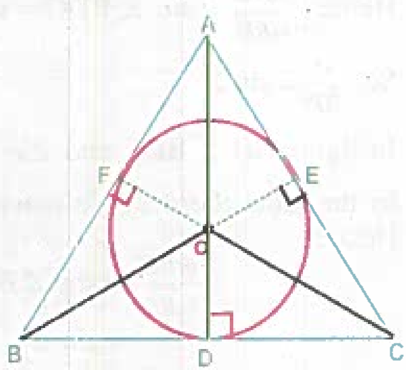


Figure 11.21

Area Δ ABC = Area Δ AOB + Area Δ BOC + Area Δ AOC

$$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = r\frac{(a+b+c)}{2} = rS$$

To obtain r the radius of inscribed circle, we divide both sides by S

Hence $\quad r = \dfrac{Area\ \Delta ABC}{S}$

If we write Δ for the area of triangle ABC then

$$r = \frac{\Delta}{S} = \frac{\sqrt{S(S-a)(S-b)(S-c)}}{S}$$

$$= \sqrt{\frac{(S-a)(S-b)(S-c)}{S}}$$

Example 20: Find the radius of the circle inscribed in a triangle whose sides are 7, 24 and 25.

Solution: We must first calculate the half perimeter S.

$$S = \frac{a+b+c}{2} = \frac{7+24+25}{2} = \frac{56}{2} = 28$$

Then $\quad r = \sqrt{\dfrac{(28-7)(28-24)(28-25)}{28}} = \sqrt{\dfrac{21\times4\times3}{28}} = \sqrt{9} = 3$

Example 21: Prove that in any triangle ABC $\; r = 4R\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}$

Solution: R.H.S $\; = 4R\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}$

$$= \frac{4(abc)}{4\Delta}\sqrt{\frac{(S-b)(S-c)}{bc}} \times \sqrt{\frac{(S-a)(S-c)}{ac}} \times \sqrt{\frac{(S-a)(S-b)}{ab}}$$

$$= \frac{1}{\Delta}(a\,b\,c)\sqrt{\frac{(S-a)^2(S-b)^2(S-c)^2}{a^2b^2c^2}} = \frac{1}{\Delta}(a\,b\,c)\times\frac{(S-a)(S-b)(S-c)}{abc}$$

$$= \frac{1}{S\Delta}\times S(S-a)(S-b)(S-c) = \frac{1}{S\Delta}\times\Delta^2 = \frac{\Delta}{S} = r$$

as $\quad \Delta^2 = S(S-a)(S-b)(S-c)$.

(c) To find the Radius of e-circle of a triangle

Let O be the e-center opposite to the vertex A as shown in Figure 11.22

Let L, M and N be the points at which the e-circle touches the side $\overline{BC}$ externally and touches the sides AB, AC when produced respectively.

Then from elementary geometry OL, OM and ON are perpendiculars to the side $\overline{BC}$ and sides AB, AC

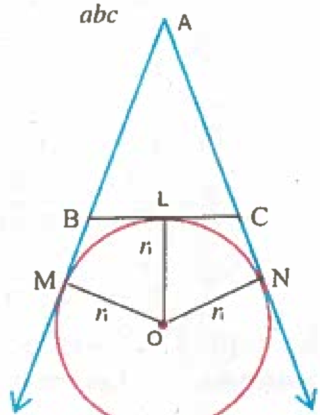


Figure 11.22

(when produced) respectively. Join the e-center O with A, B and C.

Clearly $m\overline{OL} = m\overline{OM} = m\overline{ON} = r_1$

Area Δ ABC = Area Δ AOB + Area Δ AOC – Area Δ BOC

$$= \frac{1}{2}c\,r_1 + \frac{1}{2}b\,r_1 - \frac{1}{2}a\,r_1 = \frac{1}{2}r_1(c+b-a)$$

$$= \frac{1}{2}r_1(2S-2a) \text{ where } S = \frac{a+b+c}{2}$$

Thus the area Δ of triangle ABC is

$$\Delta = r_1(S-a) \text{ or } \boxed{r_1 = \frac{\Delta}{S-a}}$$

Similarly $\boxed{r_2 = \frac{\Delta}{S-b}}$ if the e-circle touches side b directly but sides a, c when produced.

The e-radius r_3 of escribed circle associated with vertex C is given by $\boxed{r_3 = \frac{\Delta}{S-c}}$.

Example 22: Find R, r, r_1, r_2 and r_3 for the triangle with measures of the sides 5, 12 and 13.

Solution:

Let $a = 5$, b = 12 and c = 13

$$S = \frac{1}{2}(5+12+13) = 15$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{15\times10\times3\times2} = 30$$

$$R = \frac{(abc)}{4\Delta} = \frac{5\times12\times13}{4\times30} = 6.5$$

$$r = \frac{\Delta}{S} = \frac{30}{15} = 2$$

$$r_1 = \frac{\Delta}{S-a} = \frac{30}{15-5} = 3$$

$$r_2 = \frac{\Delta}{S-b} = \frac{30}{15-12} = 10$$

$$r_3 = \frac{\Delta}{S-c} = \frac{30}{15-13} = 15$$

Example 23: Prove that for any equilateral triangle $r : R : r_1 = 1 : 2 : 3$

Solution: Let the measure of each side of the triangle be denoted by c.

$$\therefore\ S = \frac{c+c+c}{2} = \frac{3c}{2}$$

Area of the triangle is given by

$$\Delta = \sqrt{S(S-c)^3} = \sqrt{\frac{3c}{2}\left(\frac{3c}{2}-c\right)^3} = \sqrt{\frac{3c}{2}\times\frac{c^3}{8}} = \frac{\sqrt{3}c^2}{4}$$

$$R = \frac{a.b.c}{4\Delta} = \frac{c^3}{4\frac{\sqrt{3}}{4}c^2} = \frac{c}{\sqrt{3}}$$

$$r = \frac{\Delta}{S} = \frac{\sqrt{3}c^2}{4}\times\frac{2}{3c} = \frac{c}{2\sqrt{3}}$$

Now $r_1 = \dfrac{\Delta}{S-a} = \dfrac{\frac{\sqrt{3}}{4}c^2}{\frac{3c}{2}-c} = \dfrac{\sqrt{3}c^2}{4}\times\dfrac{2}{c} = \dfrac{\sqrt{3}c}{2}$

Hence, $r : R : r_1 = \dfrac{c}{2\sqrt{3}} : \dfrac{c}{\sqrt{3}} : \dfrac{\sqrt{3}c}{2}$

$$= \frac{c}{2\sqrt{3}}\times\frac{\sqrt{3}}{c} : \frac{c}{\sqrt{3}}\times\frac{\sqrt{3}}{c} : \frac{\sqrt{3}c}{2}\times\frac{\sqrt{3}}{c} = \frac{1}{2} : 1 : \frac{3}{2} = 1:2:3$$

Example 24: Find the area of the inscribed circle of the triangle whose sides measure 7, 8 and 9 unit.

Solution: Here $S = \dfrac{7+8+9}{2} = 12$

Area of triangle with sides 7, 8 and 9.

$\Delta = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{12\times5\times4\times3} = 26.83\ unit^2$

$r = \dfrac{\Delta}{S} = \dfrac{26.83}{12} = 2.24$ unit

Area of inscribed circle $= \pi r^2 = (3.1416)(2.24)^2 = 15.76\ unit^2$

EXERCISE 11.4

1. Compute the in-radius (r) and circum-radius (R) of the triangles whose sides are given;
   (i) 3, 5, 6 (ii) 21, 20, 29
2. Find the area of the inscribed circle of the triangle with measures of the sides 55m, 25m and 70m.
3. The measures of the sides of a triangle are 20, 25 and 30 decimeter. Find the radius of the described circles
   (i) Opposite to larger side (ii) Opposite to smaller side

4. Show that (i) $\sqrt{r r_1 r_2 r_3} = \Delta$ (ii) $abc(\sin\alpha + \sin\beta + \sin\gamma) = 4\Delta s$.
(iii) $r_1 r_2 r_3 = rS^2$

5. Prove that for any triangle ABC
(i) $r_1 + r_2 + r_3 - r = 4R$ (ii) $r_1 r_2 + r_2 r_3 + r_3 r_1 = S^2$
(iii) $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$

6. Show that
(i) $r_1 = s\tan\frac{\alpha}{2}$ (ii) $r_2 = s\tan\frac{\beta}{2}$ (iii) $r_3 = s\tan\frac{\gamma}{2}$

7. The sides of a triangle are in the ratio 3:7:8. The radius of the inscribed circle is 2m. Find the sides of the triangles.

REVIEW EXCERCISE 11

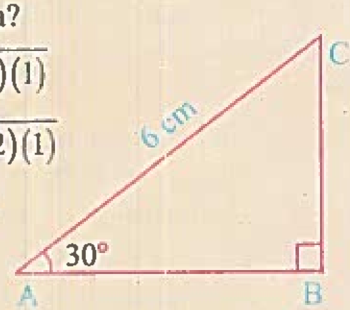
1. **Choose the correct option.**

(i) In right triangle ABC, find b if a = 2, c = 5, and r = 90^0
(a) 7 (b) 3 (c) $\sqrt{21}$ (d) $\sqrt{29}$

(ii) An escalator in a department store makes an angle of 45^0 with the ground. How long is the escalator if it carries people a vertical distance of 24 feet?
(a) $12\sqrt{2}$ ft (b) $24\sqrt{2}$ ft (c) $8\sqrt{3}$ ft (d) 48 ft

(iii) If in an isosceles triangle, 'a' is the length of the base and 'b' the length of one of the equal sides, then its area is
(a) $\frac{a}{4}\sqrt{4b^2-a^2}$ (b) $\frac{b}{4}\sqrt{4b^2-a^2}$ (c) $\frac{a+b}{4}\sqrt{a^2-b^2}$ (d) $\frac{a-b}{4}\sqrt{b^2-a^2}$

(iv) If Heron's formula is used to find the area of triangle ABC having a=3 meters, b = 5 meters, and c = 6 meters, which of the following shows the correct way to set up the formula?
(a) $\Delta = 7\sqrt{(10)(12)(13)}$ (b) $\Delta = \sqrt{(4)(2)(1)}$
(c) $\Delta = \sqrt{7(3)(5)(6)}$ (d) $\Delta = \sqrt{7(4)(2)(1)}$



(v) In the adjoining figure, the length of $\overline{BC}$ is
(a) $2\sqrt{3}$ cm (b) $3\sqrt{3}$ cm
(c) $4\sqrt{3}$ cm (d) 3 cm

(vi) If the angle of depression of an object from a 75 m high tower is 30^0, then the distance of the object from the tower is

(a) $25\sqrt{3}$ m (b) $50\sqrt{3}$ m (c) $75\sqrt{3}$ m (d) 150 m

(vii) The point of Concurrency of the right bisectors of the sides of a triangleis called

(a) In-Centre (b) Orthocenter (c) Circumcentre (d) Centroid

(viii) With usual notations $rr_1r_2r_3 =$

(a) Δ (b) Δ^2 (c) $\dfrac{abc}{\Delta}$ (d) $\dfrac{\Delta}{abc}$

2. Solve the triangles.

(i) $a = 0.7$, $c = 0.8$, $\beta = 141°30'$ (ii) $a = 34$, $b = 23$, $c = 58$

(iii) $a = 15.6$, $b = 18$, $\gamma = 35°10'$ (iv) $a = 48$, $b = 32$, $\gamma = 57°$

(v) $b = 35$, $c = 37$, $\alpha = 23°25'$ (vi) $a = 58.4$, $\beta = 37.2°$, $\gamma = 100°$

(vii) $c = 13.6$, $\alpha = 30°24'$, $\beta = 72°6'$

3. Find the measure of the smallest angle of the triangle whose sides have lengths

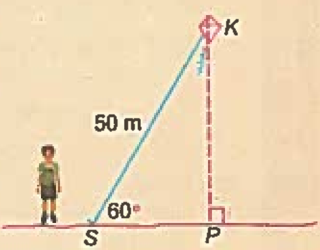
(i) 4.3, 5.1 and 6.3 (ii) 3, 4.2 and 3.8

4. Find the measure of the largest angle of the triangle whose sides have lengths

(i) 2.9, 3.3 and 4.1 (ii) 6.0, 8 and 9.4

5. The sides of a parallelogram are 25cm and 35cm long and one of its angles is 36°. Find the lengths of its diagonals.

6. A man is flying a kite. He has let out 50 m of string, and he notices that the string makes an angle of 60° with the ground. How high is the kite?



7. A robin on a branch 40ft up in a tree spots a worm at an angle of depression of 14°. From a branch 15ft above the robin, a crow spots the same worm at an angle of depression of 19°. How far is each bird from the worm?

8. The angle of elevation of a building is 48° from A and 61° from B. If AB is 20 m, find the height of the building.

UNIT 12

Graph of Trigonometric and Inverse Trigonometric Functions And Solutions of Trignomotric Equations

STUDENTS LEARNING OUTCOMES

After reading this unit, the students will be able to:

- Find the domain and range of the trigonometric functions.
- Define even and odd functions.
- Discuss the periodicity of trigonometric functions.
- Find the maximum and minimum value of a given function of the type:
  - $a+b\sin\theta$,
  - $a+b\cos\theta$,
  - $a + b\sin(c\theta + d)$,
  - $a + b\cos(c\theta + d)$,
  - the reciprocals of above, where a, b, c and d are real numbers.
- Recognize the shapes of the graphs of sine, cosine and tangent for all angles.
- Draw the graphs of the six basic trigonometric functions within the domain from -2π to 2π.
- Guess the graphs of $\sin 2\theta$, $\cos 2\theta$, $\sin \theta/2$, $\cos \theta/2$ etc without actually drawing them.
- Define periodic, even/odd and translation properties of the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$, i.e., $\sin \theta$ has
  - periodic property $\sin(\theta \pm 2\pi) = \sin \theta$,
  - odd property $\sin(-\theta) = -\sin \theta$,
  - translation property $\begin{cases} \sin(\theta-\pi) = -\sin\theta \\ \sin(\pi-\theta) = \sin\theta \end{cases}$
- Deduce $\sin(\theta + 2k\pi) = \sin \theta$ where k is an integer.
- Solve trigonometric equations of the type $\sin \theta = k$, $\cos \theta = k$ and $\tan \theta = k$, using periodic, even/odd and translation properties.

- Solve graphically the trigonometric equations of the type:
  - $\sin\theta = \theta/2$,
  - $\cos\theta = \theta$,
  - $\tan\theta = 2\theta$ when $-\pi/2 \le \theta \le \pi/2$
- Define the inverse trigonometric functions and their domain and range.
- Find domains and ranges of
  - principal trigonometric functions,
  - inverse trigonometric functions.
- Draw the graphs of inverse trigonometric functions.
- Prove the addition and subtraction formulae of inverse trigonometric functions.
- Apply addition and subtraction formulae of inverse trigonometric functions to verify related identities.
- Solve trigonometric equations and check their roots by substitution in the given trigonometric equations so as to discard extraneous roots.
- Use the periods of trigonometric functions to find the solution of general trigonometric equations.

12 Introduction

Trigonometric functions are usually defined either with the help of a unit circle or right angled triangles. We will also study their properties with a special emphasis on their graphs. Rest of the unit is concerned with inverse trigonometric functions and solutions of trigonometric equations.

12.1 Trigonometric functions

We know that the domain of the function defined by the equation $y=f(x)$ is the set of all those values of x for which the function attains finite definite values, and the range is the set of all those values which y attains. So far the functions we have studied all had subsets of real numbers as their domain and range. But the domains of trigonometric functions are the set of angles, rather than real numbers. We can however, make the domains of the trigonometric function, subsets of real numbers, by defining them on the unit circle, that is a circle whose radius is 1.

Let θ be a central angle of the unit circle and P(x, y) be the point as shown in the Figure 12.1 then $r = OP = 1 = \sqrt{x^2+y^2}$, and the six trigonometric ratios also called **trigonometric functions** or **circular functions** of θ are defined as follows:

$\text{sine}\,\theta = \frac{y}{1} = y$

$\text{cosine}\,\theta = \frac{x}{1} = x$

$\text{tangent}\,\theta = \frac{y}{x}\ (x \neq 0)$

$\text{cosecant}\,\theta = \frac{1}{y}\ (y \neq 0)$

$\text{secant}\,\theta = \frac{1}{x}\ (x \neq 0)$

$\text{cotangent}\,\theta = \frac{x}{y}\ (y \neq 0)$

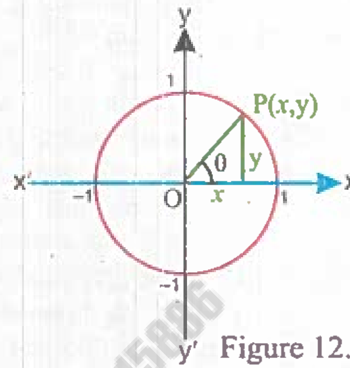


Figure 12.1

The trigonometric functions are abbreviated as follows:

(i) Sine θ as sin θ
(ii) Cosine θ as cos θ
(iii) Tangent θ as tan θ
(iv) Cosecant θ as cosec θ
(v) Secant θ as sec θ
(vi) Cotangent θ as cot θ

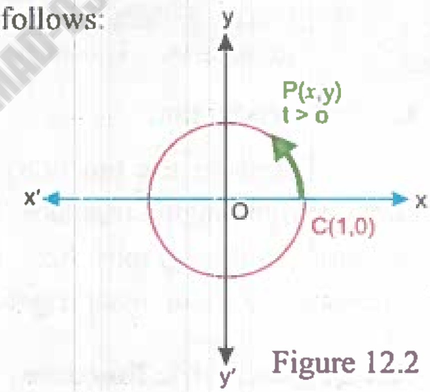


Figure 12.2

It can be seen that

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Since any real number can represent the length of exactly one Arc on the unit circle. If t is a positive number, we can find the Arc of length t by measuring a distance t in counter clockwise direction along an Arc of the unit circle beginning at C(1,0). So we get Arc CP of length t.

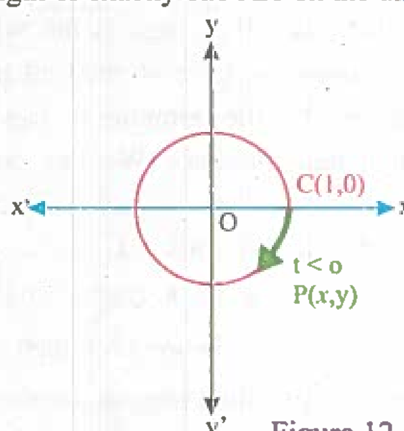


Figure 12.3

If t is a negative number, we can find the Arc of length t, by measuring a distance t in a clockwise direction along an Arc of the unit circle beginning at the point C(1,0).

In each case, we get a unique point

P(x,y) that corresponds to the real number t. We also know that if s is an Arc which subtends an angle θ at the centre of circle with radius r, we have $s=r\theta$ (θ in radians).

Let s = t and r = 1 then above equation reduces to $t=\theta$ or $\theta=t$.

Thus we obtain $\sin\theta = \sin t$, $\operatorname{cosec}\theta = \operatorname{cosec} t$

$\cos\theta = \cos t$, $\sec\theta = \sec t$

$\tan\theta = \tan t$, $\cot\theta = \cot t$.

where θ is the angle measured in radians and t is a real number.

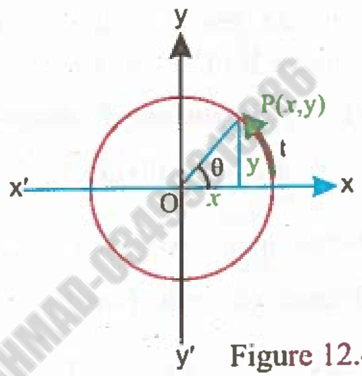


Figure 12.4

Thus we can think of each trigonometric expression as being either a trigonometric function of an angle measured in radians or as a trigonometric function of a real number t.

Thus the trigonometric functions can be thought of as functions that have domains and ranges that are subsets of real numbers.

12.1.1 Domain and Range of Trigonometric Functions

(a) Domain and Range of Sine and Cosine Functions

Refer to Figure 12.1, $\sin\theta = y$ $\cos\theta = x$

Domain of sine and cosine is the set of real numbers R. Since point P(x,y) is on the unit circle

$\therefore$ $-1 \le y \le 1$ and $-1 \le x \le 1$ or $-1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$.

Thus the range of sine and cosine functions are [–1, 1].

(b) Domain and range of tangent and cotangent functions

Refer to Figure 12.1. $\tan\theta = \dfrac{y}{x}$ $x \ne 0$.

When $x \ne 0$, then terminal side OP cannot coincide with OY or OY′; in other words

$$\theta \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\ldots \quad \text{or} \quad \theta \ne (2n+1)\frac{\pi}{2};\ n \in Z$$

Therefore for the tangent function.

$$\text{Domain} = R - \left\{ t \,\middle|\, t = (2n+1)\frac{\pi}{2};\ n \in Z \right\} \text{ and Range} = R \text{ (the set of real numbers)}$$

Since $\cot\theta = \frac{x}{y}$, $y \neq 0$.

when $y \neq 0$ then terminal side OP does not coincide with OX or OX′ in other words

$\theta \neq 0, \pm\pi, \pm 2\pi, \ldots\ldots$ or $\theta \neq n\pi, n \in Z$.

Hence in case of cotangent function Domain = $R - \{t \mid t = n\pi, n \in Z\}$

Range = R (The set of real numbers)

(c) Domain and Range of Secant and Cosecant functions

Refer to Figure.12.1. $\operatorname{cosec}\theta = \frac{1}{y}$, $y \neq 0$

If $y \neq 0$ then as seen in the case of $\cot\theta$, $\theta \neq n\pi; n \in Z$.

Domain of cosec function = $R - \{t \mid t = n\pi; n \in Z\}$.

Since $|y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2} = 1$ (Figure 12.1)

Hence $|y| \leq 1$. or $\frac{1}{|y|} \geq 1$.

Thus either $\frac{1}{y} \geq 1$ or $\frac{1}{y} \leq -1$ that is $\operatorname{cosec}\theta \geq 1$ or $\operatorname{cosec}\theta \leq -1$.

That is $\operatorname{cosec}\theta$ attains all values except those which lie between -1 and 1.

Hence Range of cosec function = $R - \{t \mid -1 < t < 1\}$.Now $\sec\theta = \frac{1}{x}$, $x \neq 0$. Then as seen in the case of $\tan\theta$. $\theta \neq (2n+1)\frac{\pi}{2}$, $n \in Z$.

Domain of secant function = $R - \{t \mid t = (2n+1)\frac{\pi}{2}, n \in Z\}$.

Also $|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} = 1$

$|x| \leq 1$ or $\frac{1}{|x|} \geq 1$.

Thus either $\frac{1}{x} \geq 1$ or $\frac{1}{x} \leq -1$ that is $\sec\theta \geq 1$ or $\sec\theta \leq -1$.

That is $\sec\theta$ attains all values except those which lie between -1 and 1.

Range of secant function = $R - \{t \mid -1 < t < 1\}$.

We now give table of the domain and range of trigonometric functions; which are written in words as well as in symbolic notations:

Function	Domain	Range
y=sin x	All real numbers; $-\infty < x < \infty$	$-1 \leq$ real numbers ≤ 1 $-1 \leq y \leq 1$
y=cos$x$	All real numbers; $-\infty < x < \infty$	$-1 \leq$ real numbers ≤ 1 $-1 \leq y \leq 1$
y=tan x	All real numbers except $(2n+1)\dfrac{\pi}{2}$, $n \in Z$. $-\infty < x < \infty$, $x \neq (2n+1)\dfrac{\pi}{2}$, $n \in Z$.	all real numbers; $-\infty < y < \infty$
y=cot x	All real numbers except $n\pi$, $n \in Z$. $-\infty < x < \infty$, $x \neq n\pi$, $n \in Z$.	all real numbers; $-\infty < y < \infty$
y=sec x	All real numbers except $(2n+1)\dfrac{\pi}{2}$, $n \in Z$. $-\infty < x < \infty$, $x \neq (2n+1)\dfrac{\pi}{2}$, $n \in Z$.	all real numbers ≤ -1 or ≥ 1 $y \geq 1$ or $y \leq -1$
y=cosec x	All real numbers except $n\pi$, $n \in Z$. $-\infty < x < \infty$, $x \neq n\pi$, $n \in Z$.	all real numbers ≤ -1 or ≥ 1 $y \geq 1$ or $y \leq -1$

Example 1: Find the domain of each of the following functions.

(i) $\sec 3x$ (ii) $\tan \dfrac{1}{5} x$ (iii) $\operatorname{cosec} \dfrac{1}{2} x$

Solution (i) We know that the domain of sec t is $-\infty < t < \infty, t \neq (2n+1)\dfrac{\pi}{2}$, $n \in Z$.

If $t=3x$, then dom sec $3x$ is $-\infty < 3x < \infty$, $3x \neq (2n+1)\dfrac{\pi}{2}$, $n \in Z$

or $\quad -\infty < x < \infty, \quad x \neq (2n+1)\dfrac{\pi}{6}$, $n \in Z$

$\therefore \quad$ Dom sec $3x = R - \{x \mid x = (2n+1)\dfrac{\pi}{6}; \quad n \in Z\}$

(ii) $\quad$ Domain tant is $\quad -\infty < t < \infty, \quad t \neq (2n+1)\dfrac{\pi}{2}, \quad n \in Z$

if $t = \frac{1}{5}x$ then dom tan $\frac{1}{5}x$ is

$-\infty < \frac{1}{5}x < \infty$, $\frac{1}{5}x \neq (2n+1)\frac{\pi}{2}$, $n \in Z$

or $-\infty < x < \infty$, $x \neq \frac{5}{2}(2n+1)\pi$, $n \in Z$.

dom tan $\frac{1}{5}x = R - \{x | x = \frac{5}{2}(2n+1)\pi, n \in Z\}$

(iii) Dom cosec t is $-\infty < t < \infty$, $t \neq n\pi, n \in Z$

Let $t = \frac{1}{2}x$ then dom cosec $\frac{1}{2}x$ is $-\infty < \frac{1}{2}x < \infty$, $\frac{1}{2}x \neq n\pi . n \in Z$

or $-\infty < x < \infty$, $x \neq 2n\pi$, $n \in Z$ $\therefore$ dom cosec $\frac{1}{2}x = R - \{x | x = 2n\pi, n \in Z\}$

Example 2: Find the range of each function.

(i) $\cos 3x$ (ii) $3\tan 2x$ (iii) $2\text{ cosec }\frac{1}{3}x$

Solution: (i) We know that for all $t \in$ dom cos t, $-1 \leq \cos t \leq 1$

Let $t = 3x$ then $-1 \leq \cos 3x \leq 1$.

Hence range cos $3x$ is the closed interval $[-1, 1]$

(ii) Since for all $t \in$ dom tan t, $-\infty < \tan t < \infty$

Let $t = 2x$ then $-\infty < \tan 2x < \infty$

Hence $-\infty < 3\tan 2x < \infty$. Thus Range of $3\tan 2x$ is R.

(iii) Since for all $t \in$ dom cosec t

cosec t ≤ -1 or cosec t ≥ 1

Let $t = \frac{1}{3}x$

Then cosec $\frac{1}{3}x \leq -1$ or cosec $\frac{1}{3}x \geq 1$.

Hence 2 cosec $\frac{1}{3}x \leq -2$ or 2 cosec $\frac{1}{3}x \geq 2$.

Hence range of 2 cosec $\frac{1}{3}x = R - \{p | -2 < p < 2\}$.

12.1.2 Even and Odd Functions

Even Functions

A function f is **even** if for every x in the domain, $f(x) = f(-x)$.

Even functions are symmetric about the y-axis. For each point (x, y) on the graph, the point $(-x, y)$ is also on the graph.

The following are the graphs of even functions.

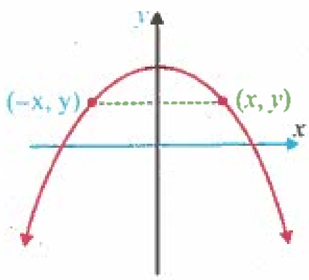


Figure 12.5

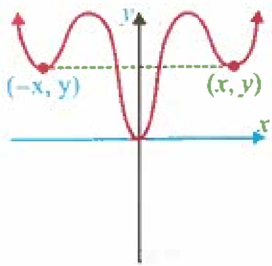


Figure 12.6

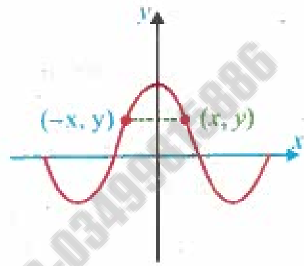


Figure 12.7

Notice that for any point (x, y) on each graph, the point $(-x, y)$ also lies on the graph. Therefore, for any x value in the domain, $f(x) = f(-x)$.

Odd Functions

A function f is **odd** if for every x in the domain, $-f(x) = f(-x)$.

Odd functions are symmetric about the origin. For each point (x, y) on the graph, the point $(-x, -y)$ is also on the graph.

The following are the graphs of odd functions.

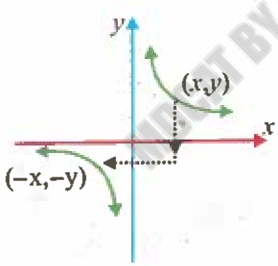


Figure 12.8

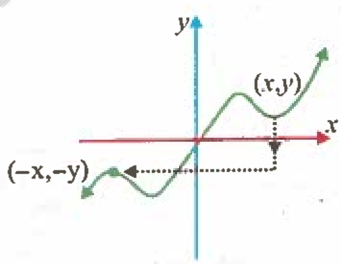


Figure 12.9

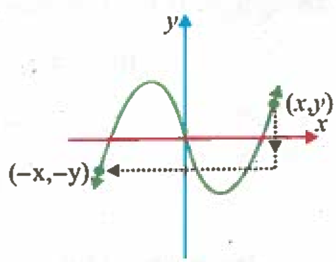


Figure 12.10

Notice that for any point (x, y) on each graph, the point $(-x, -y)$ also lies on the graph. Therefore, for any x value in the domain, $f(x) = -f(-x)$ or equivalently $-f(x) = f(-x)$.

Example 3: Prove that: (i) $f(x) = x^2$ is an even function.
(ii) $f(x) = x^3$ is an odd function.

Solution: (i) $f(x) = x^2$

$$f(-x) = (-x)^2$$
$$= x^2$$
$$= f(x)$$
$$\Rightarrow f(x) \text{ is even.}$$

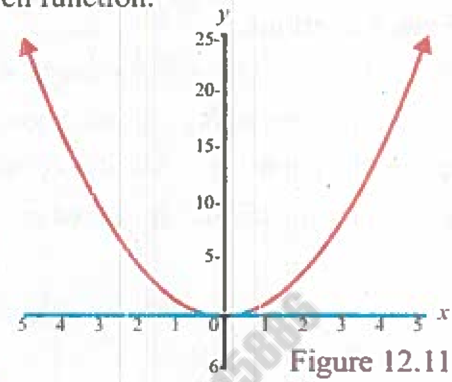


Figure 12.11

(ii) $f(x) = x^3$

$$f(-x) = (-x)^3$$
$$= -x^3$$
$$= -f(x)$$
$$\Rightarrow f(x) \text{ is odd}$$

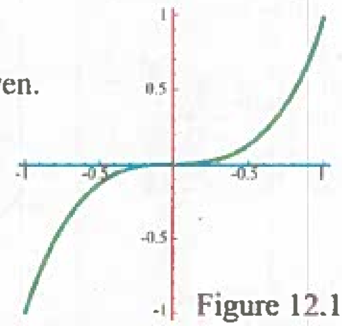


Figure 12.12

One of the important properties of the trigonometric functions is that of being either even or odd.

We know from trigonometry that:

$\sin(-\theta) = -\sin\theta,$	$\cos(-\theta) = \cos\theta,$	$\tan(-\theta) = -\tan\theta$
$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta,$	$\sec(-\theta) = \sec\theta,$	$\cot(-\theta) = -\cot\theta$

Thus sinθ, cosecθ, tanθ, cotθ are odd functions and cosθ and secθ are even functions.

Example 4: Is the function $f(x) = \sin x - \cos x$ even, odd, or neither?

$$f(-x) = \sin(-x) - \cos(-x)$$
$$= -\sin x - \cos x$$
$$= -(\sin x) + \cos x$$

Because $-(\sin x + \cos x) \neq -(\sin x - \cos x)$

And $-(\sin x + \cos x) \neq \sin x - \cos x$

the function is neither even nor odd.

Remember

The sum of an odd function and an even function is neither even nor odd.

12.1.3 Periodicity of Trigonometric Functions

Periodic Function

A function f is said to be periodic if there exists a positive constant p such that $f(x+p) = f(x)$ for all x in the domain of f. The smallest such positive number p is called the period of the function.

All the six trigonometric functions are periodic functions, because they repeat their values after their periods. This behavior of trigonometric functions is called **periodicity**.

Note

If $f(x)$ is a periodic function then $af(x)$ and $f(x) + b$ are also periodic functions and the periods of all these functions are the same. Can you say why?

Theorem 1: Show that the period of $\sin\theta$ is 2π.

Proof: If p is the period of $\sin\theta$, then

$$\sin(\theta + p) = \sin\theta \qquad (1)$$

for all $\theta \in$ dom $\sin\theta$.

Since $0 \in$ dom $\sin\theta = R$, put $\theta = 0$ in (1), we have

$\sin p = \sin 0 = 0$

Thus possible values of p are $0, \pm\pi, \pm 2\pi, \ldots\ldots$

The first smallest positive value of $p = \pi$, for which $\sin(\theta + \pi) = -\sin\theta$ which contradicts (1). Therefore π is not the period of $\sin\theta$

Next put $p = 2\pi$ then $\sin(\theta + 2\pi) = \sin\theta$

Hence 2π is the period of $\sin\theta$.

Theorem 2: Show that the period of $\cos\theta$ is 2π.

Proof: If p is the period of $\cos\theta$, then

$$\cos(\theta + p) = \cos\theta \qquad (1)$$

for all $\theta \in$ dom$\cos\theta$

Since $0 \in$ dom$\cos\theta = R$, put $\theta = 0$ in (1), we have

$\cos p = \cos 0 = 0$

Thus possible values of p are $0, \pm 2\pi, \pm 4\pi, \ldots\ldots$

The first smallest positive value of $p = 2\pi$, for which $\cos(\theta + 2\pi) = \cos\theta$.

Hence 2π is the period of $\cos\theta$.

Theorem 3: Show that the period of $\tan\theta$ is π.

Proof: If p is the period of $\tan\theta$, then

$$\tan(\theta + p) = \tan\theta \qquad (1)$$

for all $\theta \in$ dom $\tan\theta$

Put $\theta = 0$ in (1), we have

$\tan p = \tan 0 = 0$

Thus possible values of p are $0, \pm\pi, \pm 2\pi, \ldots\ldots$

The first smallest positive value of $p = \pi$,

for which $\tan(\theta+\pi) = \tan\theta$.

Hence π is the period of $\tan\theta$.

> **Remember**
>
> We may easily find out for the reciprocals of $\sin\theta$, $\cos\theta$ and $\tan\theta$, i.e. $\operatorname{cosec}\theta$, $\sec\theta$ and $\cot\theta$ that
>
> (i) 2π is the period of $\operatorname{cosec}\theta$
>
> (ii) 2π is the period of $\sec\theta$
>
> (iii) π is the period of $\cot\theta$

Example 5: Find period of $5 \sin x$.

Solution: We know that period of sine function is 2π.

$\therefore \quad \sin x = \sin(x + 2\pi)$

$\Rightarrow \quad 5\sin x = 5\sin(x + 2\pi)$

It means that when x is increased by 2π, values of $5 \sin x$ repeats, hence period of $5\sin x$ is the same as that of $\sin x$.

Thus if f is a trigonometric function, period of cf (c constant) is the same as that of f.

Example 6: Find period of $\cos 6x$.

Solution: We know that period of cosine function is 2π.

$\therefore \quad \cos 6x = \cos(6x+2\pi)$

$$= \cos 6\left(x+\frac{2\pi}{6}\right).$$

when x is increased by $\frac{2\pi}{6}$, value of $\cos 6x$ remains the same; hence period of $\cos 6x$ is $\frac{2\pi}{6}$ or $\frac{\pi}{3}$.

Thus period of $\cos 6x$ is equal to the period of $\cos x$ divided by 6.

This result holds for other trigonometric functions also

Thus, if f is a trigonometric function, then for any constant k, period of $f(kx) = \dfrac{\text{Period of } f(x)}{k}$

Example 7: Find period of each function.

(i) $\frac{1}{2}\tan 3x$ (ii) $3\sec\frac{x}{3}$

Solution: (i) Period of $\frac{1}{2}\tan 3x = \dfrac{\text{Period of } \tan x}{3}$

$$= \frac{\pi}{3} \qquad (\because \text{period of } \tan x \text{ is } \pi)$$

(ii) Period of $3\sec\frac{x}{3} = \dfrac{\text{Period of } \sec x}{\frac{1}{3}}$

$$= \frac{2\pi}{\frac{1}{3}} \qquad (\because \text{ period of } \sec x \text{ is } 2\pi)$$

$$= 3\,(2\pi) = 6\pi.$$

12.1.4 Maximum and minimum values of certain trigonometric functions

In this section we are concerned with finding the maximum and minimum value of a function of the type:

(i) $a + b\sin\theta$ (ii) $a + b\cos\theta$

(iii) $a + b\sin(c\theta + d)$ (iv) $a + b\cos(c\theta + d)$

and the reciprocals of the above, where *a*, *b*, *c* and *d* are real numbers.

Before doing so, we recall that the term *a* in the above functions allows for a vertical shift in the graph of the functions. The term *b* in the functions allows for amplitude variation of the functions.

Now to find the maximum and minimum for sine and cosine functions we only need to remember that the maximum and minimum for both $\sin\theta$ and $\cos\theta$ are 1 and –1 respectively.

Consider types (i) and (ii) above. These functions reach its maximum when both $\sin\theta$ and $\cos\theta$ are at the maximum i.e. $\sin\theta = 1$ and $\cos\theta = 1$.

So the maximum of $a + b\sin\theta = a + |b|$ (maximum of $\sin\theta$)

$$= a + |b|(1)$$

$$= a + |b| \qquad (1)$$

Similarly, the maximum of $a + b\cos\theta = a + |b|$ (2)

These functions reach its minimum when both $\sin\theta$ and $\cos\theta$ are at the minimum i.e. $\sin\theta = -i$ and $\cos\theta = -1$. So the minimum of $a + b\sin\theta = a + |b|$ (minimum of $\sin\theta$)

$= a + |b|(-1)$

$= a - |b|$ (3)

Similarly, the minimum of $a + b\cos\theta = a - |b|$ (4)

Consider types (iii) and (iv). In these types the values of c and d do not matter because they do not affect the amplitude of the function, so we treat these two types in similar way as (i) and (ii).

So the maximum value of $a + b\sin(c\theta + d) = a + |b|$ (5)

and the maximum value of $a + b\cos(c\theta + d) = a + |b|$ (6)

The minimum value of $a + b\sin(c\theta + d) = a - |b|$ (7)

and the minimum value of $a + b\sin(c\theta + d) = a - |b|$ (8)

Thus, we conclude that, if M and m respectively denote the maximum value and minimum value of the function, then we have the following formulas.

$M = a + |b|$ and $m = a - |b|$

Let M' and m' be the maximum value and minimum value of the reciprocals of the above functions, then clearly for $m > 0$, $M > 0$ and $m < 0$, $M < 0$

$M' = \frac{1}{m}$ and $m' = \frac{1}{M}$

and for $m < 0$, $M > 0$

$M' = \frac{1}{M}$ and $m' = \frac{1}{m}$

Example 8: Find maximum and minimum values of the functions.

(i) $y = 1 + 2\sin\theta$ (ii) $y = 3 + 2\cos(3\theta - 2)$ (iii) $y = \dfrac{1}{1 + 3\sin(2\theta - 15)}$

Solution: (i) Here $a = 1$ and $b = 2$

$\therefore$ the maximum value of $y = M = a + |b|$

$= 1 + |2| = 1 + 2 = 3$

and the minimum value of $y = m = a - |b|$

$= 1 - |2| = 1 - 2 = -1$

(ii) Here $a = 3$ and $b = 2$

$\therefore M = a + |b| = 3 + |2| = 5$ and $m = a - |b| = 3 - |2| = 1$

(iii) Let $y' = 1 + 3\sin(2\theta - 15)$

Then $M = 1+|3| = 4$ and $m = 1-|3| = -2$

If M' and m' are the maximum value and minimum value of y respectively, then

$$M' = \frac{1}{M} = \frac{1}{4} \quad \text{and } m' = \frac{1}{m} = \frac{1}{-2}$$

Since $m < 0$ and $M > 0$

EXERCISE 12.1

1. Find the domain, range and period of each of the following function:

(i) $3 \sin 3x$ (ii) $\tan \frac{1}{2}x$ (iii) $\operatorname{cosec} 2x$

(iv) $\cos 4x$ (v) $6 \sec 2x$ (vi) $\frac{7}{2} \cot \frac{2\pi x}{3}$

(vii) $-\frac{1}{4} \tan x$ (viii) $\frac{1}{2} \operatorname{cosec} x$ (ix) $\sec \frac{\pi}{4} x$

2. Find maximum and minimum of each of the following functions:

(i) $y = -2 + \frac{1}{2} \sin\left(\frac{1}{3}\theta + 2\right)$ (ii) $y = 5 - 4\sin(\theta + 30)$

(iii) $y = \frac{1}{19 - 10\sin(3\theta - 45)}$ (iv) $y = \frac{1}{4\cos 2\pi\theta}$

12.2 Graphs of Trigonometric Functions

The graph of a real valued function is the set of points in the cartesian plane, whose co-ordinates are the ordered pairs, belonging to the given function. For example to graph a function y=f(x), we give a number of values to x, which belong to the domain of the function, and find the corresponding values of y, which satisfy the equation y=f(x). We plot these ordered pairs (x, y), join them by smooth curves or line segments, the diagram so formed is the graph of the function.

In case of trigonometric functions the points are joined by smooth curves. Since trigonometric functions are periodic, it is sufficient to draw graph over a period. This information can be used to extend the graph to the right and the left, because the graph will be identical over those values of x which form the period.

12.2.1 The shapes of the graphs of sine, cosine and tangent

$y = \sin x$ and **$y = \cos x$** look pretty similar; in fact the main difference is that the sine graph starts at (0,0) and the cosine at (0,1).

Both of these graphs repeat every 360 degrees, and the cosine graph is essentially a transformation of the sin graph—it's been translated along the x-axis by 90 degrees. Thinking about the fact that $\sin x = \cos(90 - x)$ and $\cos x = \sin(90 - x)$, **$y = \tan x$** crosses the x-axis at 0, and has an asymptote at 90. This graph repeats every 180 degrees.

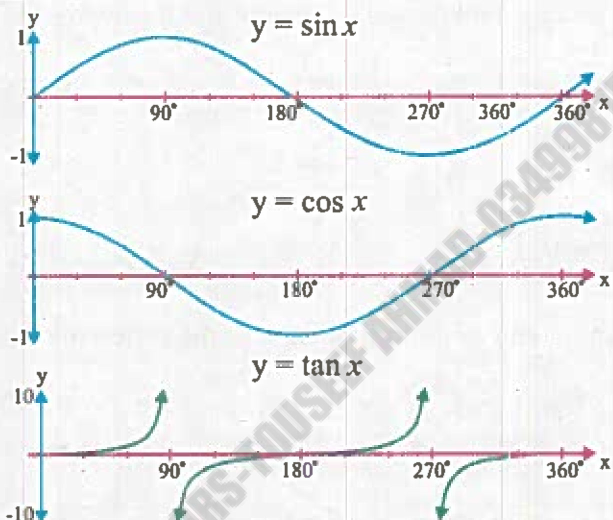


Figure 12.13

12.2.2 Graphs of six basic trigonometric functions

(a) Graph of $y = \sin x, -2\pi \le x \le 2\pi$

Since $\sin x$ is periodic function of period 2π, whose domain is R, it is sufficient to draw a detailed graph over the interval $[0, 2\pi]$; portions x of the graph over the intervals $[-2\pi, 0]$, $[0, 2\pi]$, $[2\pi, 4\pi]$ and so on will be identical.

Suitable values of x, and the corresponding values of y, satisfying $y = \sin x$ are given below in the form of a table.

Values of y for different angles x, can be found by use of trigonometric identities.

x	0 0°	$\frac{\pi}{6}$ 30°	$\frac{\pi}{3}$ 60°	$\frac{\pi}{2}$ 90°	$\frac{2\pi}{3}$ 120°	$\frac{5\pi}{6}$ 150°	π 180°	$\frac{7\pi}{6}$ 210°	$\frac{4\pi}{3}$ 240°	$\frac{3\pi}{2}$ 270°	$\frac{5\pi}{3}$ 300°	$\frac{11\pi}{6}$ 330°	2π 360°
y	0	0.5	0.87	1	0.87	0.5	0	0.5	-0.87	1	-0.87	0.5	0

Take a set of rectangular axes, choosing a convenient length for 30° on the x–axis and a convenient length as a unit on the y–axis. We plot the points (x, y) to get the following graph of $y=\sin x$ in the interval $(0, 2\pi)$

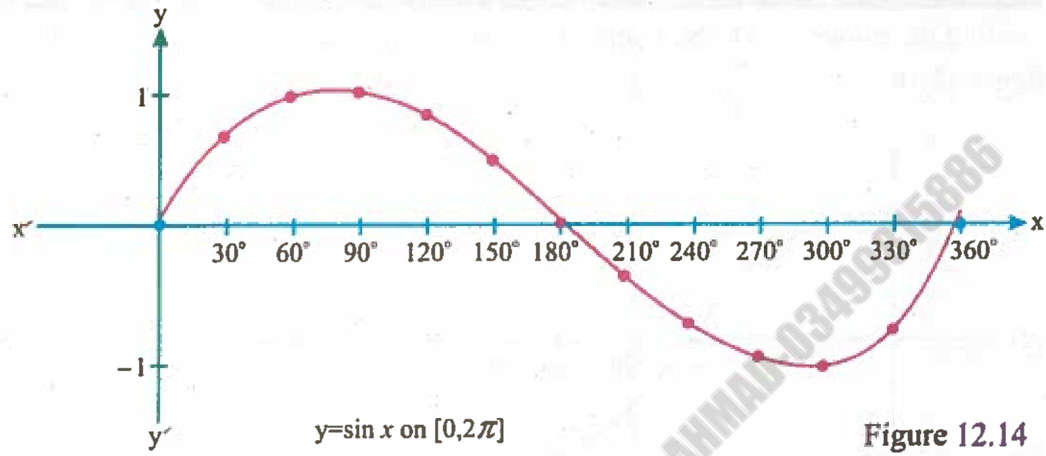


Figure 12.14

We note for all values of x, $-1 \leq \sin x \leq 1$.

We often call the graph of $y = \sin x$, a sine wave and the graph in the interval $[0,2\pi]$ a cycle. Extended graph of $\sin x$ which is the repetition of the graph in figure 12.14 is given in figure 12.15.

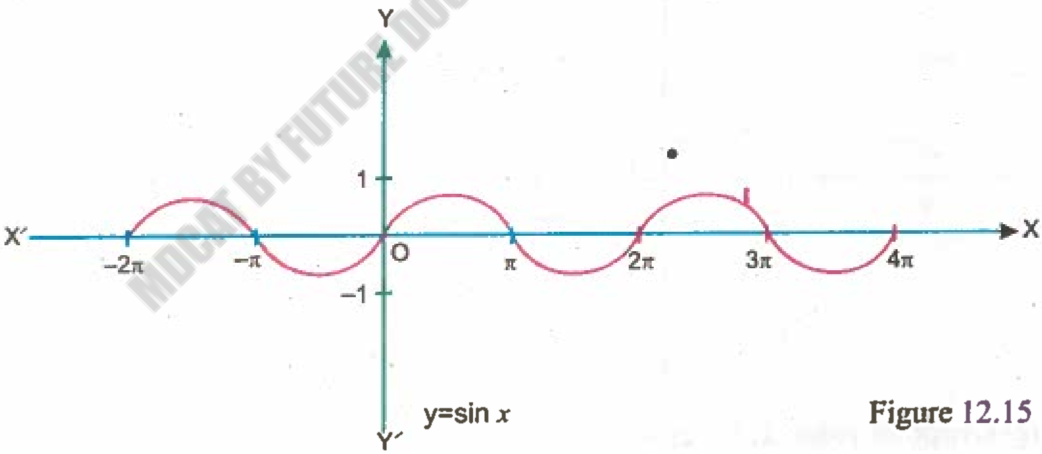


Figure 12.15

(b) Graph of $y = \text{Cos}\, x$, $-2\pi \leq x \leq 2\pi$.

The cosine function also has a period of 2π, and its range is $[-1,1]$. Values of (x, y), satisfying $y=\cos x$ are given below in the form of a table.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1

Plotting the points (x, y); the graph of y=cos$x$ on the interval $[0, 2\pi]$, is shown in figure 12.16.

Figure 12.16

Extended graph of y=cos$x$ is shown in figure 12.17.

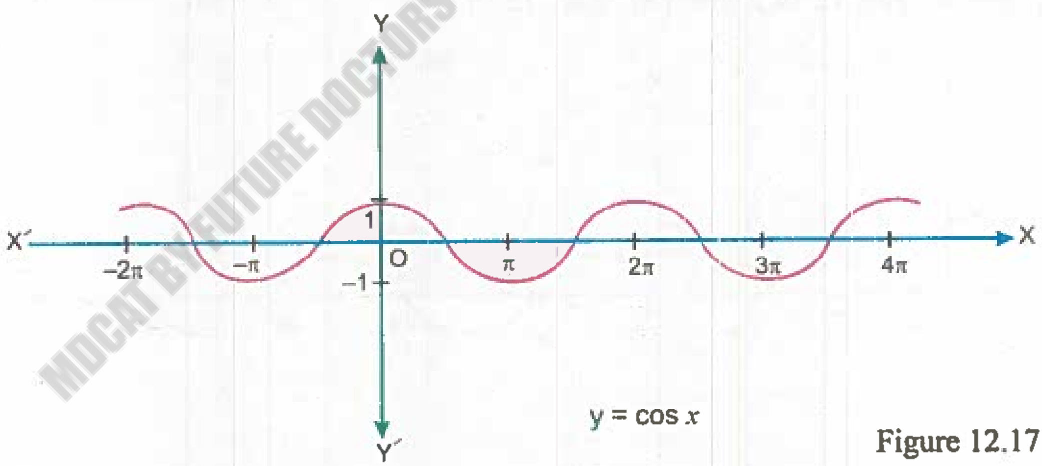


Figure 12.17

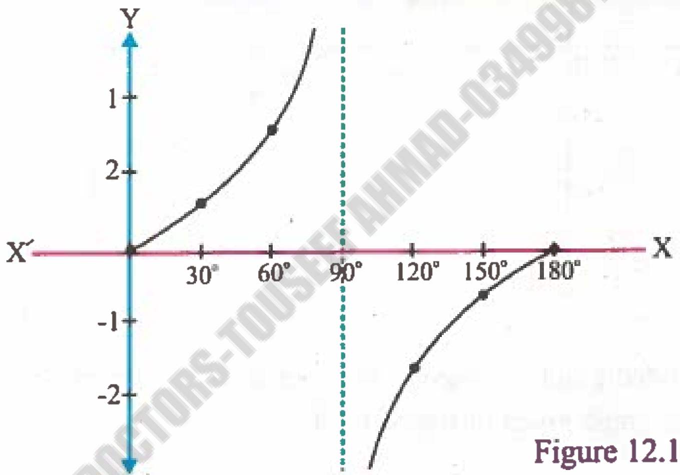
(c) Graph of y=tan x, $0 \le x \le \pi$

The period of tan x is π and the domain is the set $R-\{x \mid x=(2n+1)\frac{\pi}{2}, n \in Z\}$.

When $x=(2n+1)\frac{\pi}{2}$, $n \in Z$ or $x = \pm 90°, \pm 270°, \ldots\ldots$; the tangent function is not defined, at these values of x, it becomes very large, in other words it approaches $\pm\infty$.

In the interval $[0, \pi]$ as we approach 90° from the left, the tangent becomes larger positively, that is, it tends to $+\infty$; and when we approach 90° from the right, it becomes larger negatively, that is it tends to $-\infty$. Table of values (x, y) satisfying y=tan x on $[0, \pi]$ are given in the below table. The graph is shown in figure 12.18.

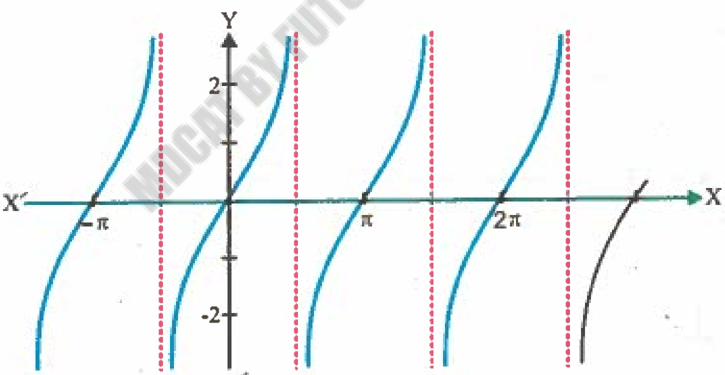
x	0°	30°	60°	90°	120°	150°	180°
y	0	0.58	1.73	$-\infty$ ∞	−1.73	−0.58	0°



y = tan x on [0, 2π]

Figure 12.18

Extended graph of y = tan x is given in figure 12.19.



y= tan x

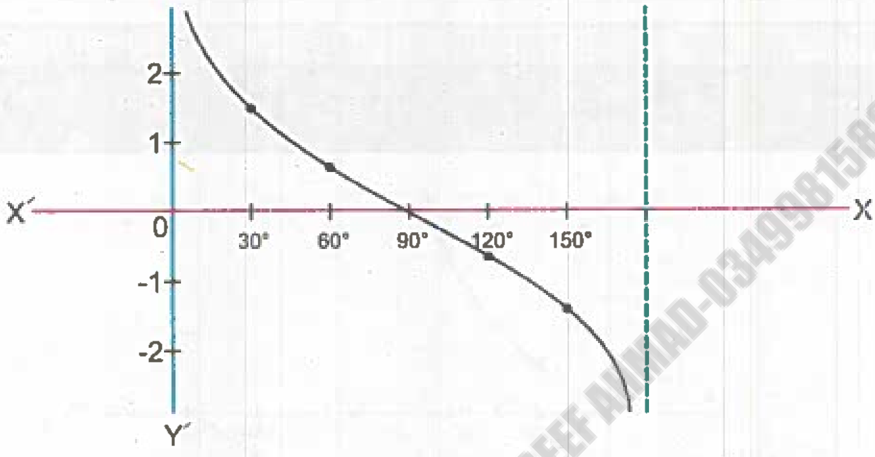
Figure 12.19

(d) Graph of y=cot x, $-\pi \le x \le \pi$.

The period of cotangent is also π.

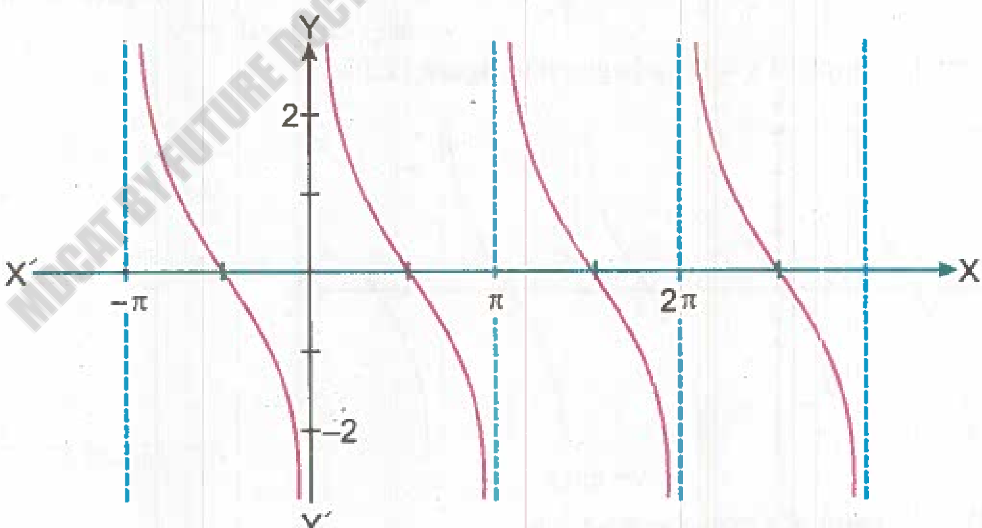
Table of values (*x*, y) satisfying y=cot *x* on [0, π] is given below, while graph is shown in figure 12.20.

x	0°	30°	60°	90°	120°	150°	180°
y	∞	1.73	0.58	0	−0.58	−1.73	$-\infty$



y = cot*x* on [0,π]   Figure 12.20

Extended graph of y=cot *x* is given below in figure 12.21, which is the repetition of the graph given in figure 12.20.

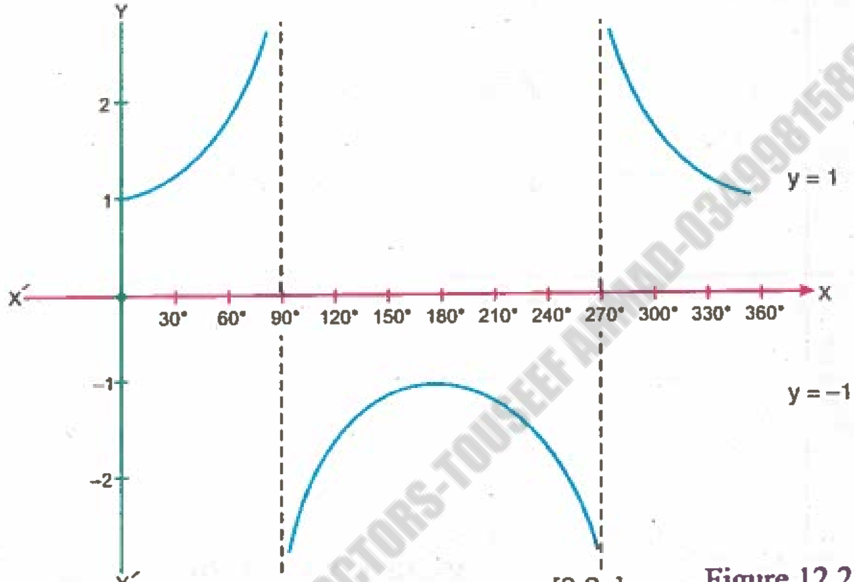


y = cot *x*   Figure 12.21

(e) Graph of $y = \sec x, -2\pi \le x \le 2\pi$.

We know that period of secant is 2π, table of values (x, y) for $y=\sec x$ on $[0, 2\pi]$ is given below. Graph is shown in figure 12.22.

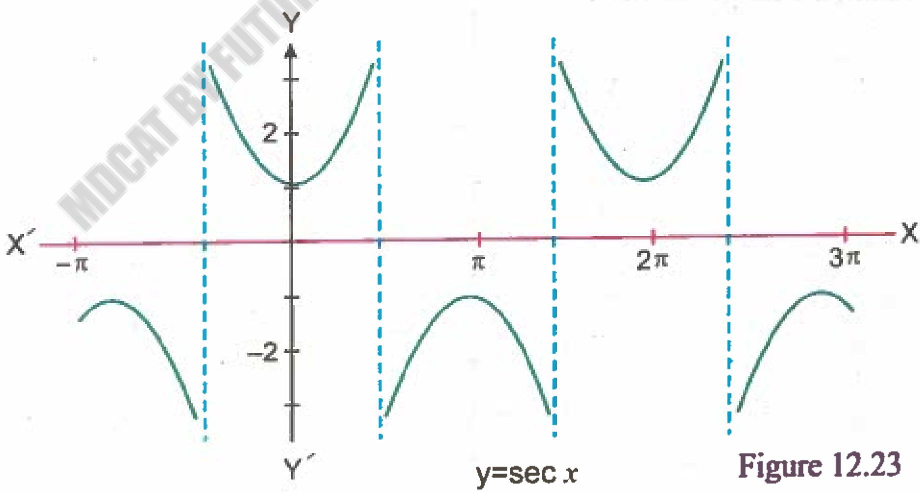
x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y	1	1.15	2	∞	−2	−1.15	−1	−1.15	−2	−∞	2	1.15	1



y=sec x on [0,2π]

Figure 12.22

Extended graph of y=sec x



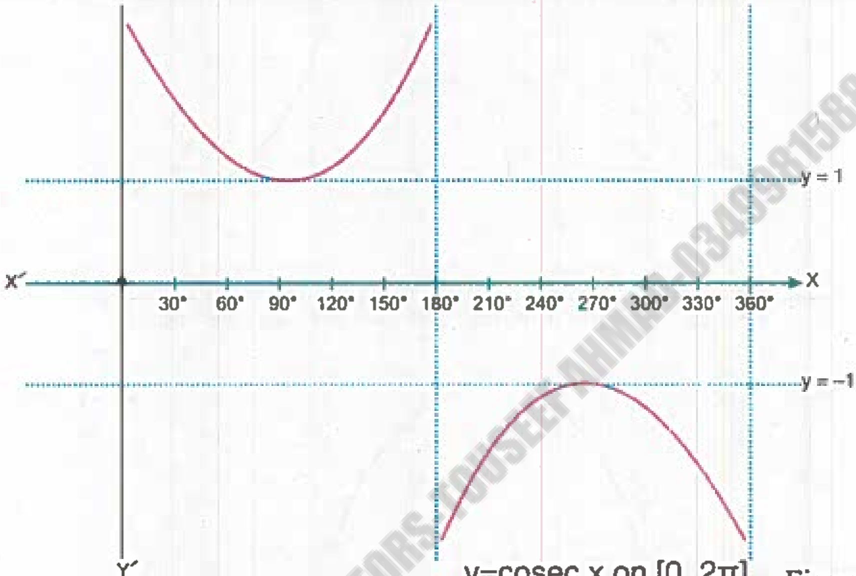
y=sec x

Figure 12.23

(f) Graph of $y = \operatorname{cosec} x, \quad -2\pi \le x \le 2\pi$

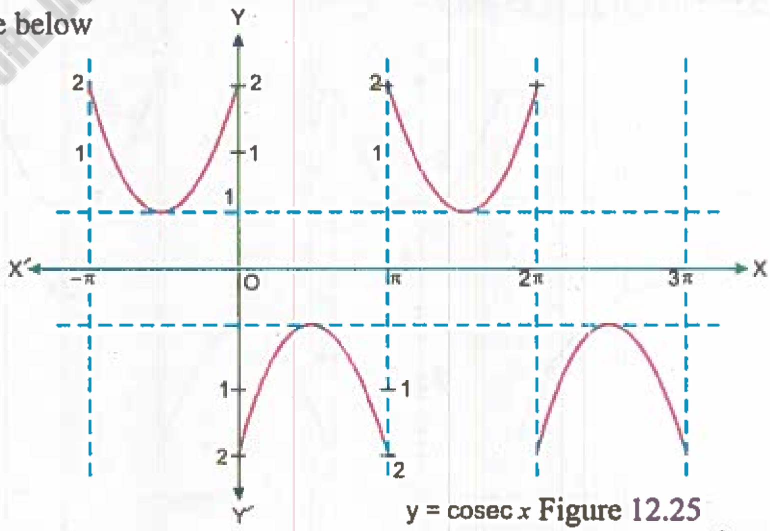
Period of cosec is 2π, table of values (x, y) satisfying $y = \operatorname{cosec} x$ on $[0, 2\pi]$ is as follows:

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y	∞	2	1.15	1	1.15	2	$-\infty$	−2	−1.15	−1	−1.15	−2	∞



y=cosec x on [0, 2π] Figure 12.24

Repeating the graph in figure.12.24, the extended graph of $y = \operatorname{cosec} x$ is obtained as given in the figure below



$y = \operatorname{cosec} x$ Figure 12.25

12.2.3 Graphs of sinAθ and cosAθ where A is a positive constant.

In figure 12.26 the graph of $y = \sin\theta$ is shown.

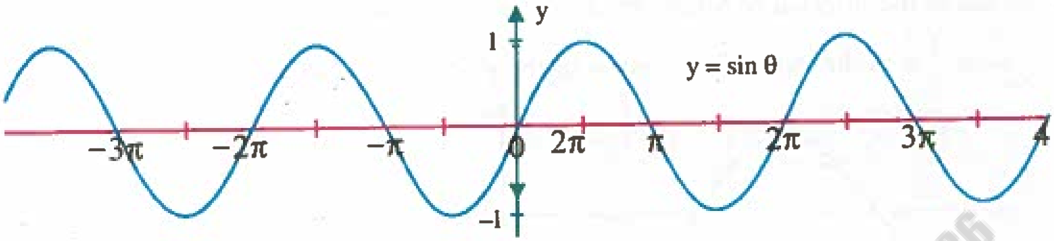


Figure 12.26

We see that the graph of $y = \sin\theta$ has period 2π, so the constant A in $y = \sin A\theta$ indicates the number of periods in the interval of length 2π. If $y = \sin\theta$, we notice that $A = 1$. This means that there is only 1 period in that interval. For example, if $A = 2$, then

$$y = \sin 2\theta$$

means that there are 2 periods in an interval of length 2π as shown in figure 12.27. The graph of $y = \sin 2\theta$ is the compressed version of the graph of $y = \sin\theta$ in the x–direction.

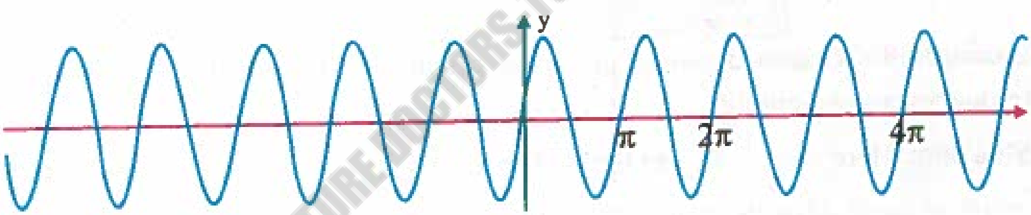


Figure 12.27

If $A = 3$, then $y = \sin 3\theta$ indicates that there are 3 periods in the interval of length 2π as shown in figure 12.28. The graph of $y = \sin 3\theta$ is more compressed version of the graph of $y = \sin\theta$ as compared to the graph of $y = \sin 2\theta$.

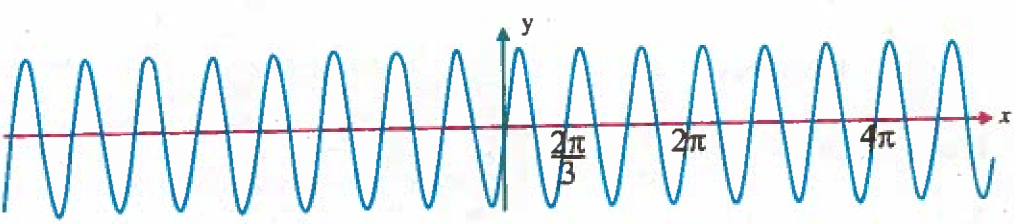


Figure 12.28

On the other hand, if $A = \frac{1}{2}$, then $y = \sin\frac{1}{2}\theta$ means that there is only half a period in the interval of length 2π as shown in **figure 12.29.** The graph of $y = \sin\frac{1}{2}\theta$ is the expanded version of the graph of $y = \sin\theta$ in the x–direction.

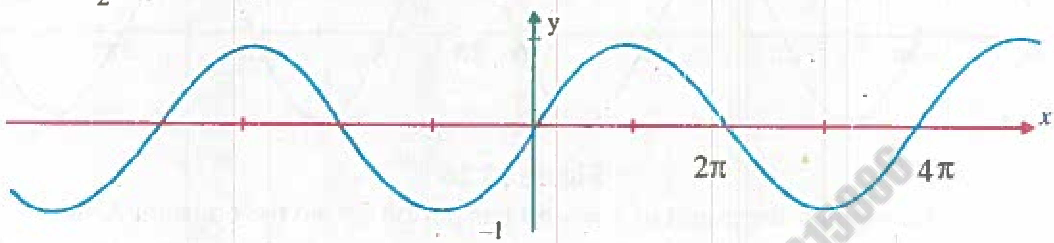


Figure 12.29

Similarly, multiplying θ by a positive constant has the geometric effect of compressing or expanding the graph of y = cosθ in the x–direction.

Thus, multiplying θ by a number greater than 1 compresses the graph of sinθ or cosθ in the x–direction and shortens its period. Multiplying θ by a positive number less than 1 expands the graph and lengthens its period. In this case the period is given by $\boxed{\text{Period} = \frac{2\pi}{A}}$

Example 8: Without drawing, guess the graph of $\cos\frac{1}{3}\theta$. Also find its period, frequency and amplitude.

Solution: Here $A = \frac{1}{3} < 1$, so the graph of $\cos\frac{1}{3}\theta$ is an expanded version of the graph of cosθ. Also in an interval of length 2π, there is one third of a period.

We have $\quad \text{Period} = \frac{2\pi}{A}$

$\therefore \quad \text{Period of } \cos\frac{1}{3}\theta = \frac{2\pi}{\frac{1}{3}}\theta = 6\pi$

$\text{Frequency} = \frac{A}{2\pi}$

$\therefore \text{ Frequency of } \cos\frac{1}{3}\theta = \frac{1}{6\pi}$

$\text{Amplitude of } \cos\frac{1}{3}\theta = 1$

Did You Know ?

(i) The period is also called the **wave length.**

(ii) The reciprocal of the period is called the frequency of the functions. Thus

$\boxed{\text{Frequency} = \frac{A}{2\pi}}$

(iii) The maximum distance between the graph of the sine or cosine and the horizontal axis is called the **amplitude** of the function. Thus, the functions y = sinθ and y = cosθ have amplitude 1. In general, the amplitude of a periodic function is half of the difference between the maximum and minimum values.

12.2.4 Periodic, Even/Odd and Translation Properties of the Graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$

In section 12.2, we draw the graphs of all six trigonometric functions. If we examine the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$, we observe that they have many symmetry properties.

In this sections we are concerned with the periodic, even/odd and translation properties of the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

1. Symmetry properties of the graph of Sin θ

The graph of $\sin\theta$ is reproduced in figure 12.30.

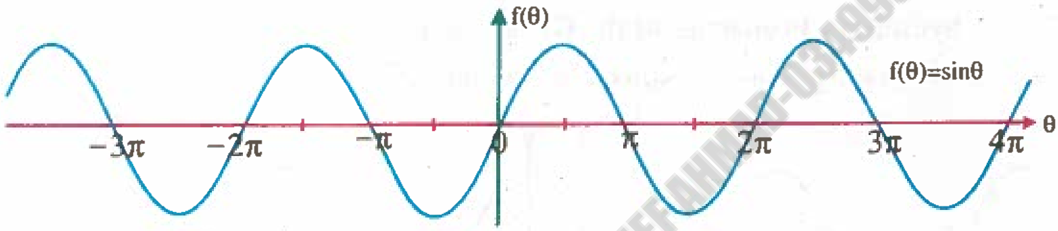


Figure 12.30

(a) Periodic Properties

We see that the graph of $\sin\theta$ keeps repeating itself after a period of 2π units. Therefore

$$\sin(\theta \pm 2\pi) = \sin\theta$$

This property possessing by $\sin\theta$ is called the **periodic property**.

(b) Even/Odd Property

The graph $\sin\theta$ is symmetrical about the origin. This means that if we replace θ by $-\theta$, the graph is changed. Therefore

$$\sin(-\theta) = -\sin\theta$$

This shows that $\sin\theta$ is an odd function which is in conformity with the results in theorem of section 12.1.2. This property possessing by $\sin\theta$ is called the **odd property**.

(c) Translation Property

If in figure 12.30, θ is decreased or increased by π, then the sign of $f(\theta)$ is changed. Therefore

$$\sin(\theta - \pi) = -\sin\theta$$

This property possessing by $\sin\theta$ is called the **translation property**.

Using the odd and translation properties, we have

$$\sin(\pi-\theta) = \sin[-(\theta-\pi)] = -\sin(\theta-\pi) = -(-\sin\theta) = \sin\theta$$

i.e. $\sin(\pi-\theta) = \sin\theta$

Thus, the graph of $\sin\theta$ possesses the following properties:

- Periodic property: $\sin(\theta \pm 2\pi) = \sin\theta$
- Odd Property: $\sin(-\theta) = -\sin\theta$
- Translation Property: $\begin{cases} \sin(\theta-\pi) = -\sin\theta \\ \sin(\pi-\theta) = \sin\theta \end{cases}$

2. Symmetry Properties of the Graphs of $\cos\theta$

The graph of $\cos\theta$ is reproduced in figure 12.31.

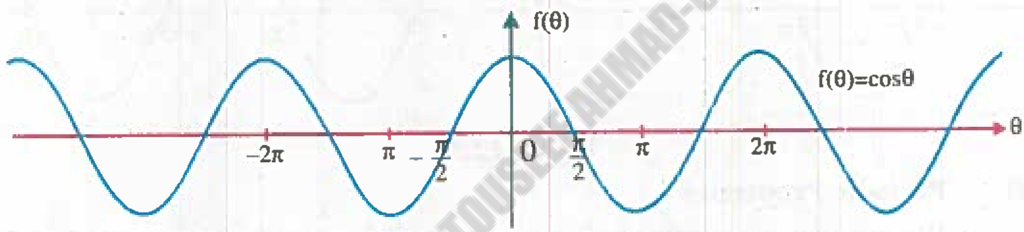


Figure 12.31

(a) Periodic Properties

Like $\sin\theta$, the graph of $\cos\theta$ also repeats itself after a period of 2π. Therefore

$$\cos(\theta \pm 2\pi) = \cos\theta$$

This property possessing by $\cos\theta$ is called the **periodic property**.

(b) Even/Odd Property

The graph of $\cos\theta$ is symmetrical about the y–axis. This means that if we replace θ by $-\theta$, the graph is unchanged. Therefore $\cos(-\theta) = \cos\theta$

This shows that $\cos\theta$ is an even function which is also in conformity with the results in theorem of section 12.1.2. This property possessing by $\cos\theta$ is called the **even property**.

(c) Translation Property

If in figure 12.31, θ is decreased or increased by π unit, then the sign of $f(\theta)$ is changed. Therefore $\cos(\theta-\pi) = -\cos\theta$

This property possessing by $\cos\theta$ is called the **translation property**.

Also $\cos(\pi-\theta) = \cos[-(\pi-\theta)] = \cos(\pi-\theta) = -\cos\theta$

i.e. $\cos(\pi-\theta) = -\cos\theta$

Thus, the graph of $\cos\theta$ possesses the following properties:

- Periodic property: $\cos(\theta \pm 2\pi) = \cos\theta$
- Even Property: $\cos(-\theta) = \cos\theta$
- Translation Property: $\begin{cases} \cos(\theta-\pi) = -\cos\theta \\ \cos(\pi-\theta) = -\cos\theta \end{cases}$

Symmetry properties of the graph of tanθ

The graph of tanθ is shown in Figure 12.32.

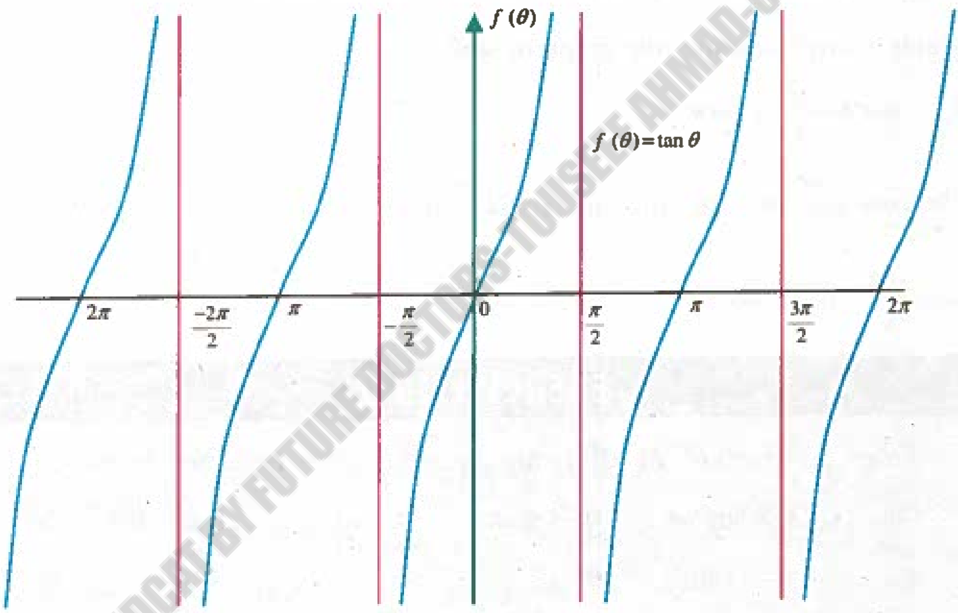


Figure 12.32

The symmetry properties of the graph of tanθ can be obtained in similar fashion as in the case of sinθ and cosθ. However, it is pertinent to note that in the present case the period of tanθ is π. Therefore, the translation property of the graph of tanθ equals its periodic property.

The properties of the graph of tanθ are given as below:

- Periodic Property: $\tan(\theta \pm \pi) = \tan\theta$
- Odd Property: $\tan(-\theta) = -\tan\theta$
- Translation Property: $\begin{cases} \tan(\theta - \pi) = \tan\theta \\ \tan(\pi - \theta) = -\tan\theta \end{cases}$

Example 9: Use the symmetric and periodic properties of the cosine, to establish the following identity. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

Solution:

By translating the graph of cosθ by $\frac{\pi}{2}$ units in the direction of the positive θ–axis the graph of cosθ becomes the graph of sinθ

That is $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$

But the cosine is an even function, so $\cos\left(\frac{\pi}{2} - \theta\right) = \cos\left(\theta - \frac{\pi}{2}\right)$

Thus, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

EXERCISE 12.2

1. Draw the graph of the following functions in the indicated interval.

(i). $y = 2\sin x$ $0 \le x \le 2\pi$ (ii). $y = \cos 2x$ $0 \le x \le 2\pi$

(iii) $y = -4 + \sin x$ $0 \le x \le \pi$ (iv) $y = -\cot x$ $-\pi \le x \le \pi$

(v) $y = 2\,\text{cosec}\,2x$ $0 \le x \le 2\pi$ (vi) $y = \sec\frac{x}{2}$ $\pi \le x \le 2\pi$

2. Without drawing, guess the graph of each of the following functions. Also find its period, frequency and amplitude.

(i) $y = \cos 2\theta$ (ii) $y = \sin 6\theta$

(iii) $y = \sin \pi\theta$ (iv) $y = \cos\frac{\pi}{2}\theta$

3. Use the symmetric and periodic properties of the sine, cosine and tangent functions to establish the following identities.

(i) $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$ (ii) $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$

(iii) $\sin(\pi-\theta) = \sin\theta$ (iv) $\cos(\pi-\theta) = -\cos\theta$

(v) $\tan(\pi-\theta) = -\tan\theta$ (vi) $\tan(2\pi-\theta) = -\tan\theta$

4. For any integer k, deduce that

(i) $\sin(\theta+2k\pi) = \sin\theta$ (ii) $\cos(\theta+2k\pi) = \cos\theta$

(iii) $\tan(\theta+2k\pi) = \tan\theta$ (iv) $\cot(\theta+2k\pi) = \cot\theta$

(v) $\sec(\theta+2k\pi) = \sec\theta$ (vi) $cosec(\theta+2k\pi) = cosec\theta$

12.3 Solution/graphical solution of trigonometric equations

An equation involving trigonometric functions is called **a trigonometric equation.**

There is no general procedure for solving all trigonometric equations. However, we can solve many trigonometric equations by means of algebraic methods such as rearranging equations, factoring, squaring and taking roots and by using the basic trigonometric identities already proved in earlier units.

12.3.1 Solution of trigonometric functions of the type sinθ = k, cosθ = k and tanθ = k

The simplest trigonometric equations are of the form

$$\sin\theta = k \qquad (1)$$
$$\cos\theta = k \qquad (2)$$
$$\tan\theta = k \qquad (3)$$

where k is a constant.

Did You Know

An **identity** is an equation which is true for all values of the variable.

In this section, we are concerned with to solve these equations, using periodic, even/odd and translation properties.

In section 12.1.3, we noticed that the sine functions and cosine functions are periodic and both have period 2π, i.e. they repeat their values every 2π units. Thus, if we want to find all solutions of (1) and (2) then we simply add and subtract integer multiple of 2π to the solutions in the interval $0 \leq \theta < 2\pi$. We also noticed that tangent function is also periodic having π as its period. Thus to find all solutions of equation (3), we add and subtract integer multiple of π to the solutions in the interval $0 \le \theta < \pi$.

Thus, to find all solutions of such trigonometric equations, first of all find the solution over the interval whose length is equal to its periods and then find the formula for all solutions of the equations.

Example 11: Solve the equation $\sin\theta = \frac{1}{2}$.

Solution: We have $\sin\frac{\pi}{6} = \frac{1}{2}$, so the reference angle is $\theta = \frac{\pi}{6}$. Since sine is positive in quadrant I and quadrant II, so the equation has two solutions in the interval $0 \le \theta < 2\pi$, one in quadrant I and the other in quadrant II i.e.

$$\theta = \frac{\pi}{6} \quad or \quad \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Now to find all solutions of the equation, we add and subtract integer multiples of 2π to the solutions $\frac{\pi}{6}$ or $\frac{5\pi}{6}$.

$$\therefore \quad \theta = \frac{\pi}{6}, \frac{\pi}{6}+2\pi, \frac{\pi}{6}-2\pi, \frac{\pi}{6}+4\pi, \frac{\pi}{6}-4\pi, \ldots\ldots\ldots$$

$$or \quad \theta = \frac{5\pi}{6}, \frac{5\pi}{6}+2\pi, \frac{5\pi}{6}-2\pi, \frac{5\pi}{6}+4\pi, \frac{5\pi}{6}-4\pi, \ldots\ldots\ldots$$

These solutions can be written compactly as follows:

$$\theta = \frac{\pi}{6}+2\pi\, n \quad or \quad \theta = \frac{5\pi}{6}+2\pi\, n \;\; for\; n = 0, \pm1, \pm2, \ldots\ldots\ldots$$

Example 12: Solve the equation $\cos\theta = \frac{1}{2}$.

Solution: We have $\cos\frac{\pi}{3} = \frac{1}{2}$, so the reference angle is $\theta = \frac{\pi}{3}$. Thus, the equation has two solutions in the interval $0 \le \theta < 2\pi$, one in quadrant I and the other in quadrant II i.e. $\theta = \frac{\pi}{3}$ or $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

To find all solutions of the equation, we add and subtract integer multiples of 2π to the solutions $\frac{\pi}{3}$ or $\frac{5\pi}{3}$.

$\therefore \qquad \theta = \frac{\pi}{3}, \frac{\pi}{3}+2\pi, \frac{\pi}{3}-2\pi, \frac{\pi}{3}+4\pi, \frac{\pi}{3}-4\pi, ...$

$or \qquad \theta = \frac{5\pi}{3}, \frac{5\pi}{3}+2\pi, \frac{5\pi}{3}-2\pi, \frac{5\pi}{3}+4\pi, \frac{5\pi}{3}-4\pi, ...$

Thus, $\theta = \frac{\pi}{3}+2\pi n$ *or* $\theta = \frac{5\pi}{3}+2\pi n$ *for* $n = 0, \pm 1, \pm 2, ...$ are all solutions of the equation.

Example 13: Solve the equation $\tan\theta = -\frac{\sqrt{3}}{3}$.

Solution: We have $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$, so the reference angle is $\theta = -\frac{\pi}{6}$. The tanθ is negative in the quadrant II and quadrant IV, however, in the interval $0 \le \theta < \pi$, the equation has one solution in the quadrant II i.e. $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Thus, all solutions of the equation are given by

$\theta = \frac{5\pi}{6}+2\pi n$ for $n = 0, \pm 1, \pm 2, ...$

12.3.2 Graphical Solution of some Trigonometric Equations

Recall that the graph of a function is the set of all points whose coordinates satisfy that function. If the graph of two functions intersects, then the coordinates of their intersection points represent a pair of numbers which satisfy both functions. The points of intersection are called the solutions of the given functions. These facts can be used to solve trigonometric equations by graphing. In this section, however, we are concerned with the graphical solution of trigonometric equation of the type:

- $\sin\theta = \frac{\theta}{2}$
- $\cos\theta = \theta$
- $\tan\theta = 2\theta$

in the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

The method of graphical solution of such equations is illustrated through the following example.

Example 14: Use graph to find the solution of the equation $\cos\theta - \theta = 0$ in the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Solution: The equation $\cos\theta - \theta = 0$ can be written as $\cos\theta = \theta$

Let $y = \cos\theta$ and $y = \theta$

If we draw the graphs of these two functions on the same set of coordinate axes, then their intersection point (if any) must be the solution of the given equation.

We construct the tables of values of the two functions as follows:

$y = \cos\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
Y	0	0.71	1	0.71	0

$y = \theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

θ	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$
Y	−0.79	0	0.79

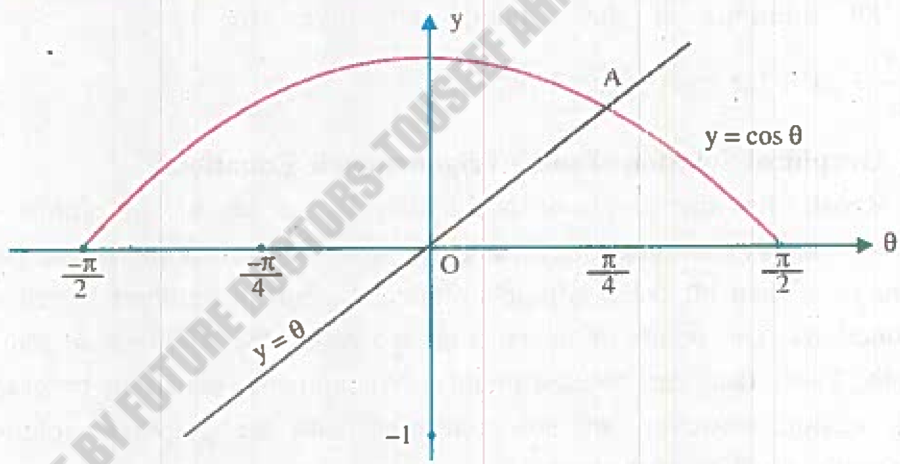


Figure 12.33

We see that the graphs intersect at point A. The point lies about midway between 0 and $\frac{\pi}{2}$. Thus, we estimate this solution as $\theta = \frac{\pi}{4}$.

Verification. Substituting the value of θ in the original equation, we obtain

$$\cos\left(\frac{\pi}{4}\right) - \frac{\pi}{4} = 0$$

$$\Rightarrow \quad 0.71 - \frac{3.14}{4} = 0 \qquad \left(\because \pi = \frac{22}{7} = 3.14 \text{ (Approx.)}\right)$$

$$\Rightarrow \quad 0.71 - 0.79 = 0$$

$$\Rightarrow \quad -0.08 = 0$$

This agreement seems quite good for a graphical approximation.

Note !

(1) If the graphs intersect at more than one point, the other solutions of the equation may similarly be estimated.

(2) We could have estimated the solution as coordinate pair (θ, y). However, the variable y does not appear in the original equation. Hence, we are interested only in values of the angle that satisfy the equation.

(3) A process of successive trial and error with use of trigonometric tables or scientific calculator would give the x–coordinate of the intersecting point as accurately as desired.

EXERCISE 12.3

1. Find all solutions of the trigonometric functions graphically.

(i) $\sin\theta = \frac{\sqrt{2}}{2}$ (ii) $\cos\theta = -\frac{\sqrt{3}}{2}$ (iii) $\tan\theta = \sqrt{3}$

(iv) $\cos\theta = \frac{1}{2}$ (v) $\tan\theta = -1$ (vi) $\sin\theta = -\frac{1}{2}$

12.4 Inverse Trigonometric Functions

12.4.1 Inverse trigonometric functions and their domain and range

We know that if f: $x \rightarrow y$ is one to one and onto, then there exists a unique function g: $y \rightarrow x$ such that $g(y) = x$, where $x \in X$ is such that $y = f(x)$. Thus, the domain of g = range of f and range of g = domain of f. The function g is called the inverse of f and is denoted by f^{-1}.

Thus, $\qquad f(x) = y \Rightarrow f^{-1}(y) = x$

(a) The Inverse Sine Function

Reproducing the graph of the sine function

$\{(x, y) \mid y = \sin x \in R\}$

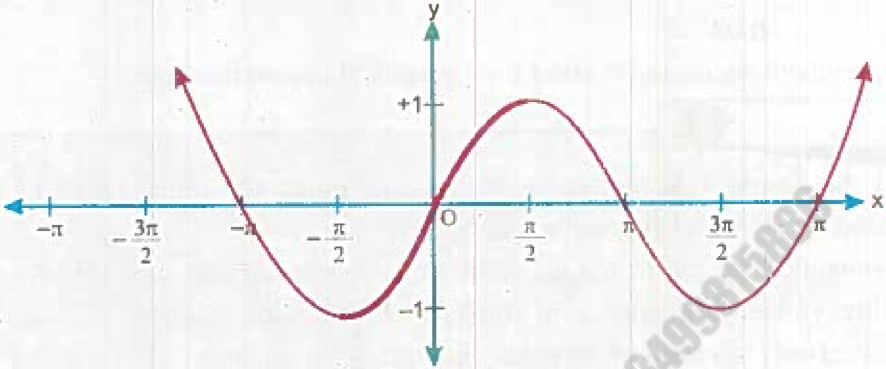


Figure 12.34

It follows from the horizontal line test that any line y = b, where b lies between –1 and +1 intersects the graph of y = sin x infinitely many times. Hence the function is not one to one. However, if we restrict the domain of y=sin x to the Interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the restricted function y = sin x, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ represented by bold portion of the curve in **Figure 12.34** is one-to-one and hence will have an inverse.

This new function with domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and range [–1, 1] is sometimes called **principal sine function** and is denoted by Sin$x$ (with capital S).

The inverse sine function denoted by Sin^{-1} is the inverse of the principal sine function and defined by:

$y = \text{Sin}^{-1}x$ if and only if $x = \text{Sin } y$, $-1 \le x \le 1$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

That graph of $y = \text{Sin}^{-1}x$ can be obtained by reflecting the restricted portion of $y = \text{Sin } x$ about the line $y = x$. The reflected graph of $y = \text{Sin}^{-1}x$ is illustrated in bold portion.

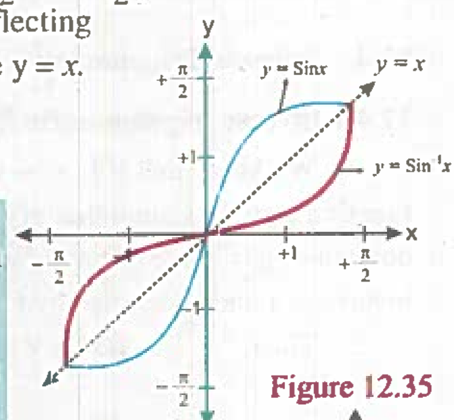


Figure 12.35

Note

Here $y = \text{Sin}^{-1}x$ means that y is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (both inclusive) whose sine is x.

The superscript –1 that appears in $y = \text{Sin}^{-1}x$ is not an exponent i.e. $\text{Sin}^{-1}x \ne \frac{1}{\text{Sin}x}$.

Inverse Relation of General Sine Functions

Generally $y = \sin^{-1}x$ gives the relation defined by

$$y = \sin^{-1}x \text{ if and only if } x = \sin y,$$

$$\text{for } -1 \leq x \leq 1,\ y \in \mathbb{R}$$

The various values obtained for a particular x represent the angles for which $x = \sin y$ and are called the inverse values of general sine functions. Since the domain of $\sin x$ is not restricted, $\sin^{-1}x$ is not itself a function. This can be proved by vertical line test.

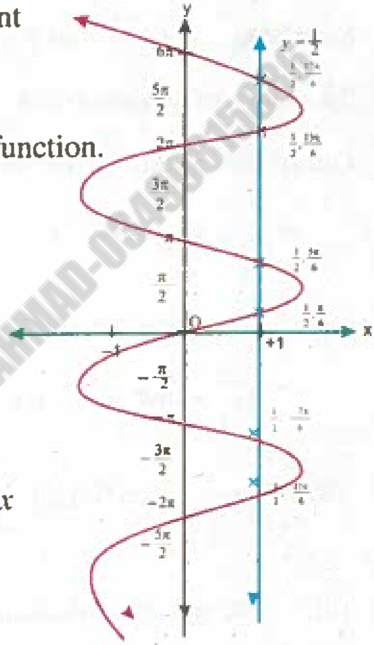


Figure 12.36

Example 15: Find the values of

(i) $\sin^{-1}(\frac{1}{2})$ (ii) $\text{Sin}^{-1}(\frac{1}{2})$

Solution (i): Figure 12.36 shows the graph of $y = \sin^{-1}x$ for $y \in R$. The line $x = \frac{1}{2}$ cuts the graph at more than one point showing that $\sin^{-1}x$ is not a function. However the intersection of $y = \sin^{-1}x$ and $x = \frac{1}{2}$ provides the various values of $\sin^{-1}(\frac{1}{2})$.

Hence from the graph in Figure 12.36 the solutions of $y = \sin^{-1}(\frac{1}{2})$ are

$$y = \frac{\pi}{6} + 2k\pi,\ k \in Z \quad \text{or} \quad y = \frac{5\pi}{6} + 2k\pi,\ k \in z$$

i.e. $y \in \{\frac{\pi}{6} + 2k\pi, k \in z\} \cup \{\frac{5\pi}{6} + 2k\pi, k \in Z\}$

(ii) Only one of the above numerous values satisfies the equation $y = \text{Sin}^{-1}(\frac{1}{2})$; that is, the value which lies in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Looking at the graph again $\frac{\pi}{6}$ is such a value. Hence $y = \text{Sin}^{-1}(\frac{1}{2}) \Rightarrow y = \frac{\pi}{6}$.

The notation Arc sin (with capital A) is sometimes used for Sin^{-1}.

Example 16: Find the exact values of the following inverse functions

(i) $\text{Sin}^{-1}(1)$ (ii) $\text{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (iii) $\text{Sin}^{-1}\left(-\frac{1}{2}\right)$

after evaluating their corresponding inverse relations.

Solution: (i) Let $y = \text{Sin}^{-1}(1)$, we seek for the general sine function sin(y), the values of y whose sine is 1. They are $\{\frac{\pi}{2}+2n\pi,\ n \in Z\}$.

Out of these values , the solution of

$$y = \text{Sin}^{-1}(1) \quad \text{is} \quad \frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

(ii) Similarly the solutions of $y = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ are

$$\left\{\frac{\pi}{3}+2n\pi,\ n\in Z\right\} \cup \left\{2\frac{\pi}{3}+2n\pi,\ n\in Z\right\}$$

However, $y = \text{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right)$, where $y\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is only $y = \frac{\pi}{3}$.

(iii) The general solutions of $y = \sin^{-1}\left(-\frac{1}{2}\right)$ are

$$y \in \left\{-\frac{\pi}{6}+2n\pi,\ n\in Z\right\} \cup \left\{-\frac{7\pi}{6}+2n\pi\right\}$$

However $y = \text{Sin}^{-1}\left(-\frac{1}{2}\right)$ yields $y = -\frac{\pi}{6}$.

Important Results. The relationship $ff^{-1}(y) = y$ and $ff^{-1}(x) = x$ that hold for every inverse functions gives us the following important results.

$$\text{Sin}^{-1}(\text{Sin}\, y) = y \quad \text{if} \ -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$\text{Sin}(\text{Sin}^{-1} x) = x \quad \text{if} \ -1 \le x \le 1$$

Example 17: Find (i) $\text{Sin}^{-1}\left[\tan\left(\frac{3\pi}{2}\right)\right]$ (ii) $\text{Sin}^{-1}\left[\tan\frac{\pi}{3}\right]$

Solution: (i) We know that $\tan\frac{3\pi}{2} = -1$

Let $y = \text{Sin}^{-1}\left[\tan\frac{3\pi}{4}\right]$, then $y = \text{Sin}^{-1}(-1)$

By definition $\text{Sin}\, y = -1$, if $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Thus y is an angle in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is -1, it follows $y = -\frac{\pi}{2}$

(ii) Let $y = \text{Sin}^{-1}\left[\tan\frac{\pi}{3}\right]$ As $\tan\frac{\pi}{3} = \sqrt{3}$ and $\sqrt{3} \notin [-1, 1]$

Hence no values of y exist which satisfies $y = \text{Sin}^{-1}(\sqrt{3})$

Thus the solution set of $y = \text{Sin}^{-1}\left[\tan\frac{\pi}{3}\right]$ is empty.

(b) The Inverse Cosine Function

In figure 12.37 we reproduce the graph of the function

$$\{(x, y) \mid y = \cos x, x \in \mathbb{R}, -1 \le y \le 1\}$$

Because every horizontal line y = b, where b lies between –1 and +1 intersects the graph of y = cos$x$ at infinitely many points, it follows that cosine function is not one–to–one.

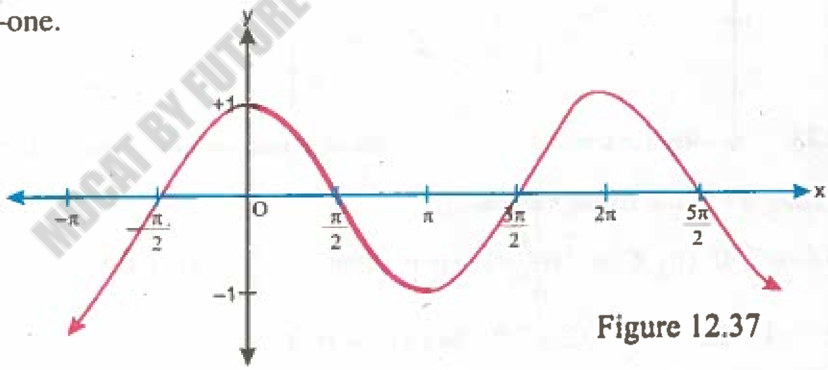


Figure 12.37

However if we restrict the domain of y = cos$x$ to the interval $[0, \pi]$ as illustrated by the bold portion of the curve in figure 12.37, we obtain a decreasing function that takes on all the values of the cosine function one and only once. This new function is called the **principal Cosine function** and is denoted by Cos$x$ (capital C).

The principal cosine function has an inverse denoted by Cos^{-1}.

The inverse cosine function Cos^{-1} is defined by

$$y = \text{Cos}^{-1} x \text{ if and only if}$$

$$x = \text{Cos}y, \text{ for } -1 \leq x \leq 1, \text{ and } 0 \leq y \leq \pi$$

This is also referred to Arc cosine function. Using general properties of inverse functions, we obtain

$$\text{Cos}(\text{Cos}^{-1} x) = \text{Cos}(\text{Arc cos}x) = x, \text{ if } -1 \leq x \leq 1,$$

$$\text{Cos}^{-1}(\text{Cos } y) = \text{Arc cos}(\text{Cos } y) = y, \text{ if } 0 \leq y \leq \pi.$$

The notation Arc cos$x$ (Capital A) is sometimes used instead of $\text{Cos}^{-1}x$.

The graph of the inverse cosine function can be found by reflecting the bold portion of Figure 12.38 in the line $y = x$. The resulting curve of $y = \text{Cos}^{-1} x$ is show in Figure 12.39 in bold portion.

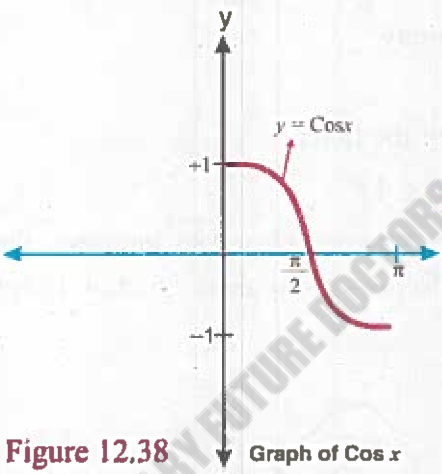
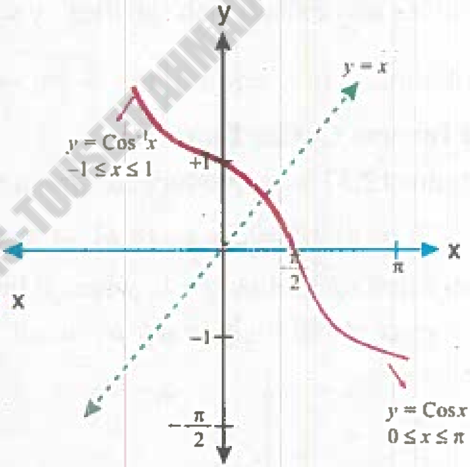


Figure 12.38 Graph of Cos x



Graph of Cos x and Arc Cos x Figure 12.39

Example 18: Find the exact values of

(i) $\text{Cos}^{-1} 0$ (ii) $\text{Cos}^{-1}(\frac{1}{\sqrt{2}})$ (iii) $\text{Cos}^{-1}(-\frac{1}{2})$ (iv) $\text{Cos}^{-1}(-\frac{\sqrt{3}}{2})$

Solution: (i) Let $y = \text{Cos}^{-1} 0$, we know that

$y = \text{Arc cos}(0)$ if and only if

$\cos y = 0$ and $y \in [0, \pi]$

Consequently, $y = \frac{\pi}{2}$ and $\text{Arc cos}(0) = \frac{\pi}{2}$

(ii) Let $y = \text{Cos}^{-1}(\frac{1}{\sqrt{2}})$. We seek the angle $0 \le y \le \pi$, whose cosine equals $\frac{1}{\sqrt{2}}$

$$\text{Cos } y = \frac{1}{\sqrt{2}}, 0 \le y \le \pi \quad \Rightarrow \quad y = \frac{\pi}{4}$$

Thus $\text{Cos}^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$

(iii) Let $y = \text{Cos}^{-1}(-\frac{1}{2})$, we seek the angle whose cosine equals $-\frac{1}{2}$. The reference point in the first quadrant is $\frac{\pi}{3}$.

Hence for negative sine we go to II quadrant by finding supplementary angle.

$$y = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \in [0, \pi]$$

Hence $\text{Cos } y = -\frac{1}{2} \Rightarrow y = \frac{2\pi}{3}$

Thus $\text{Cos}^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$

(iv) By definition.

$$y = \text{Cos}^{-1}(-\frac{\sqrt{3}}{2}), \text{ if and only if}$$

$$\text{Cos} y = -\frac{\sqrt{3}}{2}, \text{ and } 0 \le y \le \pi$$

The reference angle (1st quadrant) is $\frac{\pi}{6}$. But for negative cosine, y lies in the second quadrant (as $0 \le y \le \pi$).

Thus $y = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Hence $\text{Cos}^{-1}(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$

Example 19: Find (i) Arc cos (Cos2) (ii) Cos (Arc cos 0.5) (iii) Arc cos (Cos4) (iv) Sin (Arc sin 2.463) (v) Arc cos (cos 4) where the angles are measured in radians.

Solution: When finding these values we must pay attention to the ranges of principal trigonometric functions and their inverse functions.

(i) Since Arc cosine and Cosine (principal) are inverse function and 2 radians is between 0 and π. Hence

Arc cos (Cos2) = 2 radians

(ii) Let θ = Arc (Cos 0.5), then

Cos θ =0.5 and by substitution

Cos (Arc cos 0.5) = Cos (θ) = 0.5

(iii) For Arc cos (cos 4), we see that cos4 (general function) has the angle 4 radians in the third quadrant and therefore cos4 is negative. The Arc cosine (inverse function) of a negative value will be a second quadrant angle.

Hence Arc cos (cos4) = Arc cos (–0.653644)

$\approx$ 2.2832 radians.

(iv) (Arc sin 2.463) is not defined, since 2.463 is not between –1 and +1.

(v) Principal function Cos4 is not defined as 4 does not lie between 0 and π . Hence Arc cos (cos 4) does not exist.

(c) The Inverse Tangent function

The graph of tangent function shows that every horizontal line intersects the graph infinitely many times,it follows that tangent function is not one–to–one.

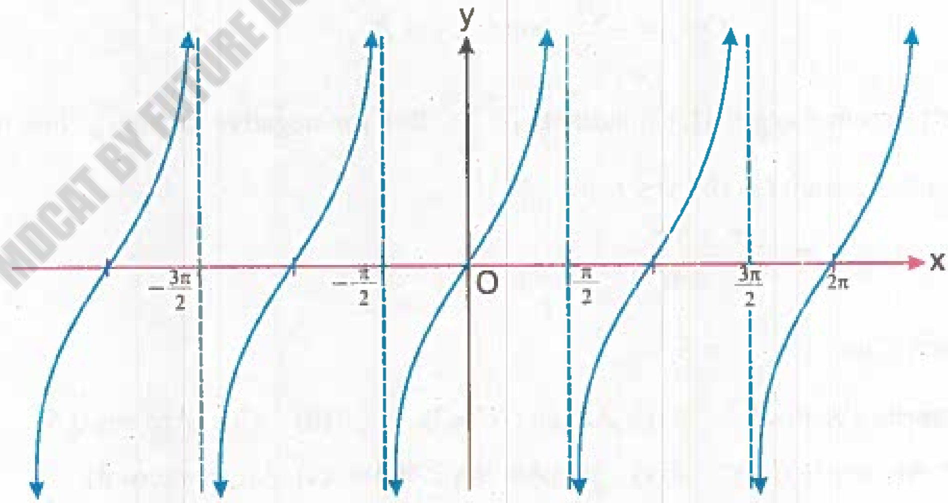


Figure 12.40

However if we restrict the domain to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, the restricted function

$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$ is one–to–one and hence has an inverse. This is called the **principal tangent function** and is denoted by $y = \text{Tan}x$ (Capital T). This leads to the definition of inverse tangent function as follows.

The inverse tangent function Tan^{-1} is defined by $y = \text{Tan}^{-1}x$ if and only if

$$x = \text{Tan}\,y, \quad \text{where } -\infty < x < \infty,\ -\frac{\pi}{2} < y < \frac{\pi}{2}$$

The graph of $\text{Tan}^{-1}x$ can be obtained as before by reflecting the principal Tangent function in the line $y = x$ as shown below:

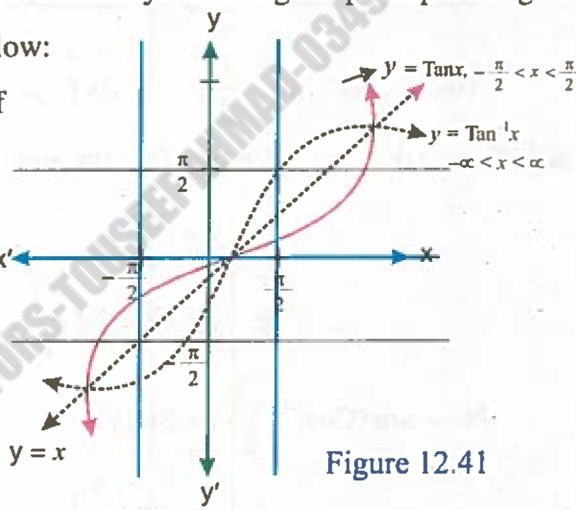


Figure 12.41

Example 20: Find the exact values of

(i) $\text{Tan}^{-1}(1)$ (ii) $\text{Tan}^{-1}(-\sqrt{3})$

(iii) $\text{Tan}(\frac{3\pi}{2})$

Solution: Let $y = \text{Tan}^{-1}(1)$.

We seek the angle y, $-\frac{\pi}{2} < y < \frac{\pi}{2}$

whose tangent equals 1, i.e.,

$\text{Tan}\,y = 1$, for $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\Rightarrow y = \frac{\pi}{4}$

Therefore $y = \text{Tan}^{-1}(1) = \frac{\pi}{4}$

(ii) Let $y = \text{Tan}^{-1}(-\sqrt{3})$. We seek the angle y where $-\frac{\pi}{2} < y < \frac{\pi}{2}$

whose tangent $-\sqrt{3}$, that is

$\text{Tan}\,y = -\sqrt{3}$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$

The reference angle in the first quadrant is $\frac{\pi}{3}$. Since $\tan(-\theta) = -\tan\theta$.

Hence $y = \text{Tan}^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

(iii) $\text{Tan}\left(\frac{3\pi}{2}\right)$ where Tan represents principal tangent function exists only for the angles in the range between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since $\frac{3\pi}{2} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Hence $\text{Tan}\left(\frac{3\pi}{2}\right)$ is not defined.

Example 21: Find the exact value of

(i) $\text{Sin}\left(\text{Cos}^{-1}\frac{\sqrt{3}}{2}\right)$ (ii) $\text{Cos}\left[\text{Tan}^{-1}(-1)\right]$

(iii) $\text{Sec}\left(\text{Sin}^{-1}\frac{1}{2}\right)$ (iv) $\cos\left[\text{Tan}^{-1}(-1)\right]$

Solution: (i) We first find the angle, $y \in [0, \pi]$ such that

$$\text{Cos}\, y = \frac{\sqrt{3}}{2}$$

$$\text{or} \quad y = \text{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right) \Rightarrow y = \frac{\pi}{6} \in [0, \pi]$$

$$\text{Now Sin}\left(\text{Cos}^{-1}\frac{\sqrt{3}}{2}\right) = \text{Sin}(y)$$

$$= \text{Sin}\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

(ii) Let $y = \text{Tan}^{-1}(-1)$. We first seek the angel $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which

$$\text{Tan}\, y = -1 \Rightarrow y = \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Now $\text{Cos}\left(\text{Tan}^{-1}(-1)\right) = \text{Cos}\,(y) = \text{Cos}\left(-\frac{\pi}{4}\right)$ is not defined because Cos is the principal Cosine function whose angle must be in the interval 0 to π.

Since $-\frac{\pi}{4} \notin [0, \pi]$, hence $\text{Cos}\left(-\frac{\pi}{4}\right)$ is not defined .

(iii) For $\text{Sec}\left(\text{Sin}^{-1}\frac{1}{2}\right)$, let $y = \text{Sin}^{-1}\frac{1}{2} \Rightarrow \text{Sin}\, y = \frac{1}{2}$, then by definition y is the angle in the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that

$$\text{Sin } y = \frac{1}{2} \Rightarrow \quad y = \frac{\pi}{6}$$

Hence $\quad \text{Sec}(\text{Sin}^{-1}\frac{1}{2}) = \text{Sec}(\frac{\pi}{6}) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

(iv) As in part (ii) $\text{Tan}^{-1}(-1) = -\frac{\pi}{4}$

Here cosθ is the general cosine function.

Hence $\cos[\text{Tan}^{-1}(-1)] = \cos(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

(d) The Remaining inverse trigonometric functions

The inverse cotangent, inverse secant and inverse cosecant are not used very widely. However, we list their definition as follows:

(i) $y = \text{Cot } x$, where $0 < x < \pi$ is called **Principal Cotangent Function** which is one–to–one and has an inverse.

$y = \text{Cot}^{-1} x$ means $x = \text{Cot } y$, where $0 < y < \pi$ and $x \in (-\infty, +\infty)$

(ii) $y = \text{Sec } x$, where $0 \le x \le \pi$, $x \ne \frac{\pi}{2}$ is called the **Principal Secant Function** which is one–to–one and has an inverse.

$y = \text{Sec}^{-1} x$ means $x = \text{Sec } y$ where

$$0 \le y \le \pi, \; y \ne \frac{\pi}{2} \text{ and } |x| \ge 1$$

(iii) $y = \text{Cosec } x$ where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $x \ne 0$ is called the **Principal Cosecant Function**, which is one–to–one and has an inverse.

$y = \text{Cosec}^{-1} x = \text{Csc}^{-1} x$ means $x = \text{Csc} y$ where, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $|x| \ge 1$

12.4.2 Domains and ranges of principal trigonometric function and inverse trigonometric functions

For convenience, the domains and ranges of principal trigonometric functions and their inverses are listed in the following table.

Functions	Domains	Range
$y = \sin x$ $y = \sin^{-1} x$	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ $-1 \le x \le 1$	$-1 \le y \le 1$ $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos x$ $y = \cos^{-1} x$	$0 \le x \le \pi$ $-1 \le x \le 1$	$-1 \le y \le 1$ $0 \le y \le \pi$
$y = \tan x$ $y = \tan^{-1} x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$ $x \in \mathbb{R}$	$y \in \mathbb{R}$ $-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \operatorname{cosec} x$ $y = \operatorname{cosec}^{-1} x$	$x \in [-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$ $x \le -1, x \ge 1$	$y \le -1, y \ge 1$ $y \in [-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$
$y = \sec x$ $y = \sec^{-1} x$	$x \in [0, \pi] - \{\frac{\pi}{2}\}$ $x \le -1, x \ge 1$	$y \le -1, y \ge 1$ $y \in [0, \pi] - \{\frac{\pi}{2}\}$
$y = \cot x$ $y = \cot^{-1} x$	$x \in (0, \pi)$ $x \in \mathbb{R}$	$y \in \mathbb{R}$ $y \in (0, \pi)$

Example 22: Evaluate: (i) Arc sec 2 (ii) Arc sec (–2) (iii) Arc tan (3.5) (iv) Arc tan (–2.3)

Solution: (i) Let θ = Arc sec2, which is an inverse function. By definition,

$\sec\theta = 2$, where $\theta \in [0,\pi] - \{\frac{\pi}{2}\}$

We Know that $\sec\frac{\pi}{3} = 2 \Rightarrow \theta = \frac{\pi}{2} \in [0, \pi]$

(ii) Let θ = Arc sec(–2). This is an inverse relation not a function. Therefore, there are infinitely many values for θ. Since secθ is negative, the reference angle lies both in the quadrants (II) and (III) which are $\theta_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\theta_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

Hence, adding the multiples of periods of sec i.e. $2\pi n$, we get

$$\left\{\theta \in \frac{2\pi}{3} + 2n\pi \ , \ \frac{4\pi}{3} + 2n\pi, \ \text{n is any integer.}\right\}$$

(iii) Arc Tan (3.5) is an inverse function whose solution must lie in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Since exact values for Arc tan(3.5) are not known, we put a calculator in radian mode, to get

$$\text{Arc tan}(3.5) \simeq 1.2925 \text{ rad.}$$

(iv) By definition of inverse tangent function $-\frac{\pi}{2} <$ Arc Tan $< \frac{\pi}{2}$. Using a calculator we have, Arc tan (–2.3) $\simeq$ –1.16

Example 23: Evaluate: (i) $\tan[\cos^{-1}(-\frac{1}{2})]$ (ii) $\tan[\text{Cos}^{-1}(-\frac{1}{2})]$

(iii) $\text{Tan}[\cos^{-1}(-\frac{1}{2})]$ (iv) $\text{Tan}[\text{Cos}^{-1}(-\frac{1}{2})]$

Solution: (i) For $\cos^{-1}(-\frac{1}{2})$, we seek an angle whose cosine is $(-\frac{1}{2})$.

The reference point is $\frac{\pi}{3}$. But cosine is negative in the II and III quadrants.

Hence the required angles are $(\pi - \frac{\pi}{3})$ and $(\pi + \frac{\pi}{3})$. Adding the period

$2n\pi$ we get, $\cos^{-1}(-\frac{1}{2}) \in \{\frac{2\pi}{3} + 2n\pi\} \cup \{\frac{4\pi}{3} + 2n\pi\}, n \in Z$

Now $\tan(\frac{2\pi}{3} + 2n\pi) = \tan(\frac{2\pi}{3}) = -\sqrt{3}$ and $\tan(\frac{4\pi}{3} + 2n\pi) = \tan(\frac{4\pi}{3}) = +\sqrt{3}$

$\therefore \tan(\cos^{-1}(-\frac{1}{2})) = \tan\frac{2\pi}{3} \cup \{\tan\frac{4\pi}{3}\} = \{-\sqrt{3}, +\sqrt{3}\}$

(ii) $\text{Cos}^{-1}(-\frac{1}{2}) = \text{Arc Cos}(-\frac{1}{2}) = 2\pi \in [0, \pi]$ But $\tan(\frac{2\pi}{3}) = -\sqrt{3}$

Therefore $\tan(\text{Cos}^{-1}(-\frac{1}{2})) = \{-\sqrt{3}\}$ only

(iii) Again $\cos^{-1}(-\frac{1}{2}) = \{\frac{2\pi}{3} + 2n\pi\} \cup \{\frac{4\pi}{3} + 2n\pi\}$

But since $\frac{2\pi}{3} \notin (-\frac{\pi}{2}, \frac{\pi}{2})$ and also $\frac{4\pi}{3} \notin (-\frac{\pi}{2}, \frac{\pi}{2})$

Neither $\frac{2\pi}{3}$ nor $\frac{4\pi}{3}$ could be the argument of principal tangent function. Thus $\text{Tan}(\cos^{-1}(-\frac{1}{2}))$ does not exist.

(iv) Similarly $\text{Tan}[\text{Arc}\cos(-\frac{1}{2})] = \text{Tan}(\frac{2\pi}{3})$ is not defined.

Example 24: Evaluate

(i) $\text{Arc}\sin(\sin\frac{12\pi}{5})$ (ii) $\text{Arc}\sin(\sin\frac{12\pi}{5})$

(ii) $\text{Arc}\cos(\cos\frac{29\pi}{7})$ (iv) $\text{Arc}\cos(\cos\frac{29\pi}{7})$

Solution: (i) As $\frac{12\pi}{5} \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$, it follows that

$$\text{Arc}\,\text{Sin}(\sin\frac{12\pi}{5}) \neq \frac{12\pi}{5}$$

However, since $\sin(\frac{12\pi}{5}) = \sin[2\pi + \frac{2\pi}{5}] = \sin\frac{2\pi}{5}$

and $\frac{2\pi}{5} \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$, we find that

$$\text{Arc}\sin(\sin\frac{12\pi}{5}) = \text{Sin}^{-1}(\sin\frac{2\pi}{5}) = \frac{2\pi}{5}$$

(ii) By definition of inverse relation of sine function

$$\text{Arc}\sin(\sin\frac{12\pi}{5}) = \frac{12\pi}{5}$$

(iii) $\text{Arc}\,\text{Cos}(\cos\frac{29\pi}{7}) \neq \frac{29\pi}{7}$,

because $\frac{29\pi}{7} \notin [0, \pi]$.

But $\cos\frac{29\pi}{7} = \cos(4\pi + \frac{\pi}{7}) = \cos\frac{\pi}{7}$ and $\frac{\pi}{7} \in [0, \pi]$.

Hence Arc $\cos(\cos\frac{29\pi}{7}) = \frac{\pi}{7}$

(iv) By definition of inverse relation of general sine function, we see that

Arc $\cos(\cos\frac{29\pi}{7})$ exists when

Arc $\cos(\cos\frac{29\pi}{7}) = \frac{29\pi}{7}$

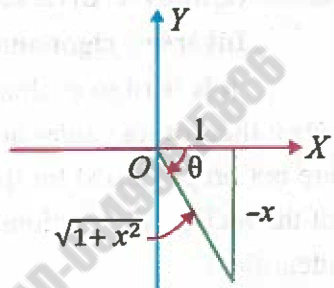


Figure 12.42

Example 25: Find the value of $\sin[\mathrm{Tan}^{-1}(-x)]$, x being a positive number

Solution: Let $\mathrm{Tan}^{-1}(-x) = \theta$. Then $\tan\theta = -x$, and θ lies between $-\pi/2$ and 0. If angle θ is constructed in standard position, as shown in Figure 12.42, then $\sin\theta$ is found to be $\frac{-x}{\sqrt{1+x^2}}$.

Hence, $\sin[\mathrm{Tan}^{-1}(-x)] = \frac{-x}{\sqrt{1+x^2}}$

EXERCISE 12.4

1. Evaluate the following inverse relations of general trigonometric functions.

(i) arc $\sin(-1)$ (ii) arc $\cos(-\frac{\sqrt{2}}{2})$ (iii) arc $\tan(-\frac{\sqrt{3}}{3})$

2. Compute the following expressions

(i) Arc $\cos\left[\tan\frac{3\pi}{4}\right]$ (ii) $Sin[Tan^{-1}(\frac{1}{\sqrt{3}})]$

(iii) $Sin\left[Arc\ \cos\left(\frac{-\sqrt{3}}{2}\right)\right]$ (iv) $\tan[Arc\ cos(\frac{-4}{5})]$

3. Find the exact value of each expression.

(i) $\mathrm{Cos}[Sin^{-1}\frac{\sqrt{2}}{2}]$ (ii) $\mathrm{Tan}[Cos^{-1}\frac{\sqrt{3}}{2}]$ (iii) $\mathrm{Sec}[Cos^{-1}\frac{1}{2}]$

(iv) $\mathrm{Cosec}[Tan^{-1}(1)]$ (v) $\mathrm{Sin}[Tan^{-1}(-1)]$ (vi) $\mathrm{Sec}[Sin^{-1}(\frac{1}{2})]$

4. Simplify the given expression, taking u as a positive number.

i) $\text{cosec}(\text{Sin}^{-1}\frac{1}{u})$ ii) $\tan(\text{Tan}^{-1}u)$

iii) $\tan(\text{Cos}^{-1}\frac{1}{\sqrt{1+u^2}})$ iv) $\text{Cos}^{-1}(\cos\sqrt{1-u^2})$

12.4.3 Graphs of Inverse trigonometric functions

Inverse Trigonometric Identities

It is hard to evaluate Arc secant, Arc cosecant, or Arc cotangent functions, when their exact values are not known, since most of the calculators or computers are not programmed for these functions. For this purpose we introduce the inverse of the reciprocal functions. The procedure is summarized by the following inverse identifies:

1. $\text{Cosec}^{-1}x = \text{Sin}^{-1}(\frac{1}{x}), x \neq 0$
2. $\text{Sec}^{-1}x = \text{Cos}^{-1}(\frac{1}{x}), x \neq 0$
3. $\text{Cot}^{-1}x = \text{Tan}^{-1}(\frac{1}{x}), x > 0$
4. $\text{Cot}^{-1}x = \pi + \text{Tan}^{-1}(\frac{1}{x}), x < 0$
5. $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1}(x)$
6. $\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1}(x)$
7. $\text{Tan}^{-1}(-x) = -\text{Tan}^{-1}(x)$
8. $\text{Sin}^{-1}(x) = \frac{\pi}{2} - \text{Cos}^{-1}(x)$

Proof. (1) Let $y = \text{Sin}^{-1}\frac{1}{x}, \; x \neq 0$ (I)

$\Rightarrow \text{Sin } y = \frac{1}{x}, \; x \neq 0 \Rightarrow \frac{1}{\text{Sin } y} = x$

$\Rightarrow \text{Cosec } y = x \Rightarrow y = \text{Cosec}^{-1}x$ (II)

From (I) and (II) $Sin^{-1}\left(\frac{1}{x}\right) = \text{Cosec}^{-1}x$

Similarly (2) can be proved.

To prove (3), let

$y = \text{Cot}^{-1}x$ where $x > 0$ (III)

$\Rightarrow y = \text{Cot}^{-1}x, \; 0 < y < \pi/2$ $\Rightarrow \text{Cot } x = y, \; 0 < y < \pi/2$

$\Rightarrow 1/\text{Tan}x = y, \; 0 < y < \pi/2$ $\Rightarrow \text{Tan}x = 1/y, \; 0 < y < \pi/2$

$\Rightarrow \quad x = \text{Tan}^{-1}(1/y)$ where $x > 0 \qquad$ (IV)

From (III) and (IV) $\qquad \text{Cot}^{-1}x = \text{Tan}^{-1}\left(\frac{1}{x}\right), x>0$

To prove (5), let

$$y = -\text{Sin}^{-1}(x) \qquad \text{(V)}$$

$$\Rightarrow \quad -y = \text{Sin}^{-1}(x) \Rightarrow \quad x = \text{Sin}(-y)$$

$$\Rightarrow \quad x = -\text{Sin}(y) \quad \Rightarrow \quad \text{Sin}\, y = -x$$

$$\Rightarrow \quad y = \text{Sin}^{-1}(-x) \qquad \text{(VI)}$$

From (V) and (VI), $\qquad \text{Sin}^{-1}(-x) = -\text{Sin}^{-1}(x)$

Similarly (6) and (7) can be proved. Finally to prove (8)

let $\theta = \frac{\pi}{2} - \text{Cos}^{-1}x \qquad$ (VII)

i.e. $\text{Cos}^{-1}x = \frac{\pi}{2} - \theta \Rightarrow x = \text{Cos}\left(\frac{\pi}{2} - \theta\right)$, for $0 \le \frac{\pi}{2} - \theta \le \pi$

Now $0 \le \frac{\pi}{2} - \theta \le \pi$ implies $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and in this range for θ, $\text{Sin}\theta$ exists.

Hence from (5) $\quad x = \text{Cos}\left(\frac{\pi}{2} - \theta\right) = \text{Sin}\,\theta$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Now $x = \text{Sin}\,\theta \Rightarrow \theta = \text{Sin}^{-1}x \qquad$ (VIII)

Substituting θ from (VIII) in (VII), we have $\quad \text{Sin}^{-1}x = \frac{\pi}{2} - \text{Cos}^{-1}x$

Example 26: Solve the equation $\quad 2\text{Sin}^{-1}x - \text{Cos}^{-1}x = \frac{\pi}{2}$

Solution: The given equation can be written as

$$2\text{Sin}^{-1}x - \text{Cos}^{-1}x + \frac{\pi}{2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

i.e. $2\,\text{Sin}^{-1}x + \text{Sin}^{-1}x = \pi \qquad \left(\because \sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x\right)$

$$or \quad 3\text{Sin}^{-1}x = \pi$$

$$\Rightarrow \quad \text{Sin}^{-1}x = \frac{\pi}{3} \quad \Rightarrow \quad x = \text{Sin}\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Example 27: Evaluate $\text{Cos}\,(\text{Sin}^{-1}\frac{4}{5} + \text{Cos}^{-1}\frac{3}{5})$ without tables or a calculator.

Solution: Let $x = \text{Sin}^{-1}\frac{4}{5}$ and $y = \text{Cos}^{-1}\frac{3}{5}$ then

$\text{Sin}\,x = \frac{4}{5}$ and $\text{Cos}\,y = \frac{3}{5}$ Where x and y are in 1st quadrant.

We have $\text{Cos}\,(x + y) = \text{Cos}\,x\,\text{Cos}\,y - \text{Sin}\,x\,\text{Sin}\,y$.

We know Sin x, but need to find Cos x, where

$$\text{Cos}\,x = \sqrt{1 - Sin^2 x}\text{, as Cos } x \text{ is +ve in 1st quadrant}$$

$$= \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

Again we know Cos y but need to find Sin y, where

$$\text{Sin}\,y = \sqrt{1 - Cos^2 y}\text{, Sin y is +ve in Quadrant I}$$

$$= \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

Therefore $\text{Cos}\left[\text{Sin}^{-1}\frac{4}{5} + \text{Cos}^{-1}\frac{3}{5}\right] = \text{Cos}\,(x + y)$

$$= \text{Cos}\,x\,\text{Cos}\,y - \text{Sin}\,x\,\text{Sin}\,y$$

$$= \frac{3}{5} \times \frac{3}{5} - \frac{4}{5} \times \frac{4}{5} = -\frac{7}{25}$$

Example 28: Evaluate $\text{Sin}(\text{Arc tan}\,\frac{1}{2} - \text{Arc cos}\,\frac{4}{5})$

Solution: Let $u = \text{Arc tan}\,\frac{1}{2}$ and $v = \text{Arc cos}\,\frac{4}{5}$,

then $\text{Tan}\,u = \frac{1}{2}$ and $\text{Cos}\,v = \frac{4}{5}$.

As Tan u is +ve, and $u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ hence u must be positive i.e.

$u \in [0, \frac{\pi}{2}]$. Similarly Cosv being positive means $v \in [0, \frac{\pi}{2}]$.

We wish to find Sin (u – v). Since u and v are in the interval

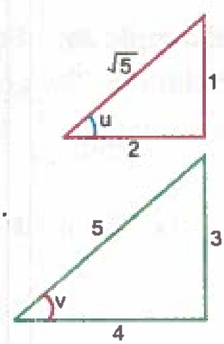


Figure 12.43

$[0, \frac{\pi}{2}]$. They can be considered as the radian measure of positive acute angles and we may construct right angled triangles for u and v as shown in fig 12.43. These triangles show that

$$\text{Sin u} = \frac{1}{\sqrt{5}} \quad , \ \text{Sin v} = \frac{3}{5}$$

$$\text{Cos u} = \frac{2}{\sqrt{5}}, \ \text{etc.}$$

Hence,

$$\text{Sin (u} - \text{v)} = \text{Sin u Cos v} - \text{Cos u Sin v}$$

$$= \frac{1}{\sqrt{5}} \times \frac{4}{5} - \frac{2}{5} \times \frac{3}{5} = \frac{4-6}{5\sqrt{5}} = \frac{-2}{5\sqrt{5}} = \frac{-2\sqrt{5}}{25}.$$

Example 29: Write (i) $\text{Cos}\ (\text{Sin}^{-1} x)$

(ii) $\text{Cos}\ (\sin^{-1} x)$ as an algebraic expression.

Solution: (i) To simplify, let $y = \text{Sin}^{-1} x$ Then $\text{Sin } y = x$ for $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

We wish to find an algebraic expression for $\text{Cos}\ (\text{Sin}^{-1} x) = \text{Cos } y$

Since $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, it follows that

$\text{Cos } y = +\sqrt{1 - Sin^2 y} = \sqrt{1-x^2}$

Consequently, $\text{Cos (Sin}^{-1}\text{x)} = \sqrt{1-x^2}$

(ii) Let $y = \sin^{-1} x, \ -1 \le x \le 1$ which is an inverse relation

$\Rightarrow x = \sin y$ is not a principal function. Hence its argument y is any real number of the set IR. Consequently. $\text{Cos } y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}$

or $\text{Cos}\ (\sin^{-1} x) = \pm\sqrt{1-x^2}$

Example 30: Express tan (Arc sin x) as an algebraic expression in x if $-1 < x < 1$.

Solution: Let $y = \text{Arc} \sin x \Rightarrow x = \text{Sin } y, \ y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Since $\tan(-\frac{\pi}{2})$ and $\tan(\frac{\pi}{2})$ are not defined, we seek to find tan y for $y \in (-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq [-\frac{\pi}{2}, \frac{\pi}{2}]$ for which $-1 \le x \le 1$

(i) If x is positive then $y \in (0, \frac{\pi}{2})$.

Figure 12.44 (i) shows the triangle for y.

(ii) If x is negative, then

$y \in (-\frac{\pi}{2}, 0)$ and the triangle for y is shown in Figure 12.44 (ii)

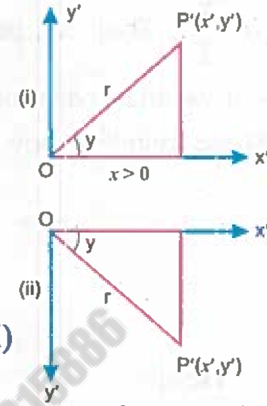


Figure 12.44

From each of the triangles, $x'^2 + y'^2 = r^2$ $\left[\sin y = \frac{y'}{r} = x'\right]$

$\Rightarrow x'^2 = r^2 - y'^2 = r^2 - r^2 x'^2 = r^2(1 - x'^2)$

$\Rightarrow \quad x' = r\sqrt{1 - x'^2}$ as x' is positive in both the cases whether

$y \in (0, \frac{\pi}{2})$ or $y \in (-\frac{\pi}{2}, 0)$

Remember

tan(Arc sin$x$) itself is positive if x is positive and negative if x is negative.

Thus $\tan y = \frac{y'}{x'} = \frac{rx}{r\sqrt{1-x^2}}$ i.e. $\tan(\text{Arc}\sin x) = \frac{x}{\sqrt{1-x^2}}$

Example 31: Verify the identify

$$\frac{1}{2}\text{Cos}^{-1}x = \text{Tan}^{-1}\sqrt{\frac{1-x}{1+x}} \quad \text{for } |x| < 1$$

Solution: Let $y = \text{Cos}^{-1}x$, we wish to show $\frac{1}{2}y = \text{Tan}^{-1}\sqrt{\frac{1-x}{1+x}}$

By half angle formula $\quad \text{Tan}\frac{y}{2} = \sqrt{\frac{1-\text{Cos}\,y}{1+\text{Cos}\,y}}$

Since $y = \text{Cos}^{-1}x$ and $|x| < 1$, it follows that $|Cos y| < 1$ and $y \in (0, \pi)$

Consequently $\frac{y}{2} \in [0, \frac{y}{2}]$ and thus $\text{Tan}\frac{y}{2} > 0$

We may drop the absolute value, obtaining $\text{Tan}\frac{y}{2} = \sqrt{\frac{1-\text{Cos}\,y}{1+\text{Cos}\,y}} = \sqrt{\frac{1-x}{1+x}}$

Thus $\frac{y}{2} = \text{Tan}^{-1}\sqrt{\frac{1-x}{1+x}}$, as required .

12.4.4 Addition and Subtraction Formulae of Inverse Trigonometric Functions

In this section we aim at to prove some important addition and subtraction formulae of inverse trigonometric functions.

1. $Sin^{-1}A + Sin^{-1}B = Sin^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)$

Let $x = Sin^{-1}A \Rightarrow Sinx = A$ and $y = Sin^{-1}B \Rightarrow Siny = B$

Since $Cos^2x + Sin^2x = 1$, so $Cosx = \pm\sqrt{1-Sin^2x} = \pm\sqrt{1-A^2}$

For Sin x = A, domain is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in which Cosine is positive, so

$Cos\,x = \sqrt{1-A^2}$. Similarly $Cos\,y = \sqrt{1-B^2}$, We have

Sin($x + y$) = Sin x Cos y + Cos x Sin y

Sin($x + y$) = Sin x Cos y + Cos x Sin y

$\Rightarrow Sin(x+y) = A\sqrt{1-A^2} + B\sqrt{1-B^2} \Rightarrow x + y = Sin^{-1}\left(A\sqrt{1-A^2} + B\sqrt{1-B^2}\right)$

$\Rightarrow Sin^{-1}A + Sin^{-1}B = Sin^{-1}\left(A\sqrt{1-A^2} - B\sqrt{1-B^2}\right)$

2. $Sin^{-1}A - Sin^{-1}B = Sin^{-1}\left(A\sqrt{1-A^2} - B\sqrt{1-B^2}\right)$

Proof of this formula is similar to (1), so is left as an exercise

3. $Cos^{-1}A + Cos^{-1}B = Cos^{-1}\left(AB - \sqrt{1-A^2}\sqrt{1-B^2}\right)$

Let $x = Cos^{-1}A \Rightarrow Cosx = A$ and $y = Cos^{-1}B \Rightarrow Cos\ y = B$

We have $\sin x = \pm\sqrt{1-Cos^2x} = \pm\sqrt{1-A^2}$

For $Cos\ x = A$, domain is $[0, \pi]$ in which Sine is positive,

So $Sin\ x = \sqrt{1-A^2}$. Similarly $Sin\ x = \sqrt{1-B^2}$

Now $Cos(x+y) = Cos\ x Cos\ y - Sin\ x\ Sin\ y$

$\Rightarrow Cos(x+y) = AB - \sqrt{1-A^2}\sqrt{1-B^2} \Rightarrow x + y = Cos^{-1}\left(AB - \sqrt{1-A^2}\sqrt{1-B^2}\right)$

$\Rightarrow Cos^{-1}A + Cos^{-1}B = Cos^{-1}\left(AB - \sqrt{1-A^2}\sqrt{1-B^2}\right)$

4. $$Cos^{-1}A - Cos^{-1}B = Cos^{-1}\left(AB + \sqrt{1-A^2}\sqrt{1-B^2}\right)$$

Proof is left as an exercise

5. $$Tan^{-1}A + Tan^{-1}B = Tan^{-1}\frac{A+B}{1-AB}$$

Let $x = Tan^{-1}A \Rightarrow Tan x = A$ and $y = Tan^{-1}B \Rightarrow Tan y = B$. We have

$$Tan(x+y) = \frac{Tan\ x + Tan\ y}{1 - Tan\,x\,Tan\ y} \Rightarrow Tan(x+y) = \frac{A+B}{1-AB} \Rightarrow Tan^{-1}A + Tan^{-1}B = Tan^{-1}\frac{A+B}{1-AB}$$

6. $$\Rightarrow Tan^{-1}A - Tan^{-1}B = Tan^{-1}\frac{A-B}{1+AB}$$

Proof is left as an exercise.

Example 32: Show that $2\,Tan^{-1}A = Tan^{-1}\frac{2A}{1-A^2}$

Solution: Put B = A in the inverse trigonometric formula

$Tan^{-1}A + Tan^{-1}B = Tan^{-1}\frac{A+B}{1-AB}$, we have

$$Tan^{-1}A + Tan^{-1}A = Tan^{-1}\frac{A+A}{1-A\cdot A} = Tan^{-1}\frac{2A}{1-A^2}$$

Example 33: Show that $Tan^{-1}\frac{3}{4} + Tan^{-1}\frac{3}{5} - Tan^{-1}\frac{8}{19} = \frac{\pi}{4}$

Solution: Using addition and subtraction formulae for $\tan^{-1}$, we have

$$Tan^{-1}\frac{3}{4} + Tan^{-1}\frac{3}{5} - Tan^{-1}\frac{8}{19} = \left(Tan^{-1}\frac{3}{4} + Tan^{-1}\frac{3}{5}\right) - Tan^{-1}\frac{8}{19}$$

$$= \left(Tan^{-1}\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right) - Tan^{-1}\frac{8}{19}$$

$$= \left(Tan^{-1}\frac{\frac{15+12}{20}}{1-\frac{9}{20}}\right) - Tan^{-1}\frac{8}{19}$$

$$=Tan^{-1}\frac{\frac{27}{20}}{\frac{11}{20}}-Tan^{-1}\frac{8}{19}=Tan^{-1}\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}$$

$$=Tan^{-1}\frac{\frac{513-88}{209}}{\frac{209+216}{209}}=Tan^{-1}\frac{425}{425}=Tan^{-1}1=\frac{\pi}{4}$$

EXERCISE 12.5

1. Find x, if

(i) $\text{Sin}^{-1}\frac{1}{2}=\frac{\pi}{2}-x$ (ii) $\text{Cos}^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{2}-\text{Sin}^{-1}x$

2. Show that

(i) $\text{Sin}^{-1}x+\text{Cos}^{-1}x=\frac{\pi}{2}$ (ii) $\text{Tan}^{-1}x+\text{Tan}^{-1}\frac{1}{x}=\frac{\pi}{2}$

(iii) $\sec(\text{Arc}\tan x)=\sqrt{1+x^2}$ (iv) $\tan(\text{Sin}^{-1}x)=\frac{x}{\sqrt{1-x^2}}$

3. Evaluate. (i) $\text{Sin}\left[\frac{\pi}{2}-\text{Cos}^{-1}\frac{4}{5}\right]$ (ii) $\text{Sin}\left[\text{Arc}\cos\frac{\pi}{2}+\pi\right]$

4. Show that (i) $\text{Cos}(\text{Sin}^{-1}x-\text{Sin}^{-1}y)=\sqrt{(1-x^2)(1-y^2)}+xy$

(ii) $\text{Cos}(2\,\text{Sin}^{-1}x)=1-2x^2,\ -1\le x\le +1$

(iii) $2\,\text{Arc Cos}\,x=\text{Arc Cos}(2x^2-1),\ 0\le x\le 1$

(iv) $\text{Cos}(\text{Arc}\tan x)=\frac{1}{\sqrt{1+x^2}}$ for $x\ge 0$

5. Express the following in terms of $\text{Tan}^{-1}(x)$

(i) $\text{Sin}^{-1}x$ (ii) Arc $\cos x$ (iii) Arc $\cot x$

6. Verify that:

(i) $2\,\text{Tan}^{-1}\left(\frac{1}{2}\right)+\text{Tan}^{-1}\left(-\frac{1}{7}\right)=\frac{\pi}{4}$

(ii) $\text{Sin}^{-1}\left(\frac{77}{85}\right)-\text{Sin}^{-1}\left(\frac{3}{5}\right)=\text{Cos}^{-1}\left(\frac{15}{17}\right)$

7. Express: $\frac{\pi}{4} - \text{Tan}^{-1}\left(\frac{1}{11}\right)$ as single inverse tangent

8. Prove that

(i) $\text{Sin}^{-1}x + \text{Cos}^{-1}x = \frac{\pi}{2}$ (ii) $\text{Tan}^{-1}x + \text{Cot}^{-1}x = \frac{\pi}{2}$

(iii) $\text{Sin}^{-1}x = \text{Tan}^{-1}\frac{x}{\sqrt{1-x^2}}$ (iv) $\text{Tan}^{-1}x = \text{Sin}^{-1}\frac{x}{\sqrt{x^2+1}}$

(v) $\text{Sin}^{-1}x = \text{Cos}^{-1}\sqrt{1-x^2}$, for $x \geq 0$ (vi) $\text{Cos}^{-1}x = \text{Tan}^{-1}\frac{\sqrt{1-x^2}}{x}$, for $x > 0$

(vii) $\text{Tan}^{-1}(x-1) + \text{Tan}^{-1}(x+1) = \text{Tan}^{-1}3x$, $(x > 0)$

12.5 Solutions of General Trigonometric Equations

Recall equations that contain trigonometric functions are called trigonometric equations. These will generally have an infinite number of solutions due to periodicity of the trigonometric function. For example the equation $\sin\theta=0$ has the solutions: $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ which can be written as: $\theta = k\pi$, where k is an integer. In a trigonometric equation, the unknown may not be the angle itself. For example in $\cos(2x+1) = 0$, the unknown is x while the angle is $(2x+1)$ and the function is cosine. We first use the definition of inverse trigonometric function to get the angle $(2x + 1)$ and then solve for x to arrive at the solution of the equation.

When a trigonometric equation contains more than one trigonometric function, trigonometric identities and algebraic formulae are used to transform such trigonometric equation to an equivalent equation that contains only one trigonometric function.

12.5.1 Techniques for Solving Trigonometric Equations

Many trigonometric equations can be solved by methods already known. The following examples illustrate by these methods.

1. **Using Factorization.**

Example 9: Solve $\tan^2 x + \sec x - 1 = 0$ in $[0, 2\pi)$

Solution: We have, $\tan^2 x + \sec x - 1 = 0$

$\text{Sec}^2 x - 1 + \sec x - 1 = 0$ using identity $1 + \tan^2 x = \sec^2 x$,
or $\tan^2 x = \sec^2 x - 1$

$\sec^2 x + \sec x - 2 = 0$

$(\sec x + 2)(\sec x - 1) = 0$ Factorizing

$\sec x = -2$ or $\sec x = 1$ Principle of zero products

$\cos x = -1/2$ or $\cos x = 1$ Using the identity $\cos x = 1/\sec x$

$x = 2\pi/3, 4\pi/3$ or $x = 0$

All these values check. The solutions in $[0, 2\pi)$ are 0, $2\pi/3$ and $4\pi/3$

Example 34: Solve $2\sin x\cos x - \sin x = 0$

Solution: $2\sin x\cos x - \sin x = 0$ (i)

$\Rightarrow$ $\sin x[2\cos x - 1] = 0$ (ii)

Equating each factor to zero, we get

$\sin x = 0$ (iii)

or $\cos x = \frac{1}{2}$ (iv)

The equation (iii) $\sin x = 0$ is satisfied by 0 and π giving the solution $\{2k_1\pi\} \cup \{2k_2\pi + \pi\}$, where $k_1, k_2 \in Z$.

This is all even multiples of π $\{2k_1\pi\}$ and odd multiples of π $\{(2k_2+1)\pi\}$ which can be simplified to $\{k\pi, k \in Z\}$.

The values of the x satisfying (iv) in the interval $[0, \pi]$ are: $\frac{\pi}{3}$ and $(2\pi - \frac{\pi}{3}) = \frac{5\pi}{3}$

Thus the solution of (iv) $\cos x = \frac{1}{2}$ is $\{\frac{\pi}{3} + 2k\pi\} \cup \{\frac{5\pi}{3} + 2k\pi\}$, $k \in Z$

Combining the two we get the general solution of the given equation (i) as

$\{k\pi\} \cup \{2k\pi + \frac{\pi}{3}\} \cup \{\frac{5\pi}{3} + 2k\pi\}$, where $k \in Z$

2. Using trigonometric identities

Example 35: Solve $4\cos^2 x + 4\sin x - 5 = 0, 0 \le x < 2\pi$

Solution: We cannot factor and solve this quadratic equation until each term involves the same trigonometric function. If we change the $\cos^2 x$ in the first term to $1 - \sin^2 x$, we will obtain an equation that involves the sine function only.

$$4\cos^2 x + 4\sin x - 5 = 0$$

$$4(1 - \sin^2 x) + 4\sin x - 5 = 0 \qquad \cos^2 x = 1 - \sin^2 x$$

$$4 - 4\sin^2 x + 4\sin x - 5 = 0 \qquad \text{Distributive property}$$

$$-4\sin^2 x + 4\sin x - 1 = 0 \qquad \text{Add 4 and } -5$$

$$4\sin^2 x - 4\sin x + 1 = 0 \qquad \text{multiply each side by } -1$$

$$(2\sin x - 1)^2 = 0 \qquad \text{Factor}$$

$$2\sin x - 1 = 0 \qquad \text{set factor to 0}$$

$$\sin x = ½$$

$$x = \pi/6,\ 5\pi/6$$

Example 36: Solve $\sin 2x \cos x + \cos 2x \sin x = \frac{1}{\sqrt{2}}$

Solution: We can simplify the left side by using the formula for sin(A+B)

$$\sin 2x \cos x + \cos 2x \sin x = \frac{1}{\sqrt{2}}$$

$$\sin(2x + x) = \frac{1}{\sqrt{2}}$$

$$\sin(3x) = \frac{1}{\sqrt{2}}$$

First we find all possible solutions for x:

$$3x = \frac{\pi}{4} + 2k\pi \quad \text{or} \quad 3x = \frac{3\pi}{4} + 2k\pi \qquad \text{k is any integer}$$

$$x = \frac{\pi}{12} + \frac{2k\pi}{3} \quad \text{or} \quad x = \frac{\pi}{4} + \frac{2k\pi}{3} \qquad \text{Divide by 3}$$

Example 37: Solve $\sin\theta - \cos\theta = 1$, if $0 \le \theta < 2\pi$

Solution: If we separte $\sin\theta - \cos\theta$ on opposite sides of the equal sign, and then square both sides of the equation, we will be able to use an identity to write the equation in terms of one trigonometric function only.

$$\sin\theta - \cos\theta = 1$$

$$\sin\theta = 1 + \cos\theta \qquad \text{Add } \cos\theta \text{ to each side}$$

$$\sin^2\theta = (1 + \cos\theta)^2 \qquad \text{Square each side}$$

$$\sin^2\theta = 1 + 2\cos\theta + \cos^2\theta \qquad \text{Expand } (1 + \cos\theta)^2$$

$$1 - \cos^2\theta = 1 + 2\cos\theta + \cos^2\theta \qquad \sin^2\theta = 1 - \cos^2\theta$$

$$0 = 2\cos\theta + 2\cos^2\theta \qquad \text{Standard form}$$

$$0 = 2\cos\theta(1 + \cos\theta) \qquad \text{Factorize}$$

$$2\cos\theta = 0 \quad \text{or} \quad 1 + \cos\theta = 0 \qquad \text{Set factors to 0}$$

$\cos\theta = 0 \quad \text{or} \quad \cos\theta = -1$

$\theta = \pi/2, 3\pi/2 \quad \text{or} \quad \theta = \pi$

We have three possible solutions, some of which may be extraneous because we squared both sides of the equation in step 2. Any time we raise both sides of an equation to an even power, we have the possibility of introducing extraneous solutions. We must check each possible solution in our original equation.

Checking $\theta = \pi/2$

$\sin\pi/2 - \cos\pi/2 = 1$

$1 - 0 = 1$

$1 = 1$, true

$\theta = \pi/2$ is a solution

Checking $\theta = \pi$

$\sin\pi - \cos\pi = 1$

$0 - (-1) = 1$

$1 = 1$, true

$\theta = \pi$ is a solution

Checking $\theta = 3\pi/2$

$\sin 3\pi/2 - \cos 3\pi/2 = 1$

$-1 - 0 = 1$

$-1 = 1$, false

$\theta = 3\pi/2$ is not a solution

3. Using Quadratic Formula

Example 38: Solve $\cos 2x = 3(\sin x - 1)$ for all eal values of x.

Solution:

$\cos 2x = 3(\sin x - 1)$ given

$1 - 2\sin^2 x = 3\sin x - 3$ double angle formula

$2\sin^2 x + 3\sin x - 4 = 0$ quadratic equation

$$\sin x = \frac{-3 \pm \sqrt{9-(4)(2)(-4)}}{(2)(2)}$$ use quadratic formula

$$\sin x = \frac{-3 \pm \sqrt{41}}{4}$$

$\sin x = -2.351$ or $0.85-8$

The first answer, -2.351, is not a solution, since the sine function must range between -1 and 1. The second answer, 0.8508, is a valid value.

$$x = \sin^{-1} 0.8508 + 2k\pi, \qquad x = \pi - \sin^{-1} 0.8505 + 2k\pi$$

In radian form,

$x = 1-0175 + 2k\pi \qquad x = 2.124 + 2k\pi$

Example 39: Find the general solution of the equation.

$$2\sin^2 x + 3\sin x - 2 = 0$$

Solution: The equation is quadratic in $\sin x$, we get

$$\sin x = \frac{-3 \pm \sqrt{9+16}}{4} = \frac{-3 \pm 5}{4}$$

$$\Rightarrow \sin x = \frac{1}{2}, -2.$$

$\sin x \in [-1,1]$, it follows that $\sin x = \frac{1}{2}$ has a solution but $\sin x = -2$ has no solution because $-2 \notin [-1,1]$.

The equation $\sin x = \frac{1}{2}$ is satisfied by the reference angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ in the interval $[0,2\pi]$. Thus the general solution set of the given equation is

$$\{\frac{\pi}{6} + 2n\pi\} \cup \{\frac{5\pi}{6} + 2n\pi\}, \text{ where } n \in Z$$

4. A Reduction Identity

Applications of inverse trigonometric functions are very useful in graphing to study the behavior of some wave functions and also in calculus and space sciences. It involves an identity to reduce the form of a trigonometric linear function i.e. $a\cos\theta + b\sin\theta = c$

where a, b, c are constants, either $a = 0$ and $b \neq 0$

Example 40: Solve the equation.

$$\sqrt{3}\cos\theta - \sin\theta = 0 \qquad \text{(i)}$$

Solution: Compare the given equation with the expression

$a\sin\theta + b\cos\theta$ we get, $a = -1$, $b = \sqrt{3}$

Let $-\sin\theta + \sqrt{3}\cos\theta = r\sin(\theta + \alpha)$ (ii)

We know that $r = \sqrt{a^2+b^2}$, $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

$$\Rightarrow \quad r = 2, \ \cos\alpha = -\frac{1}{2}, \ \sin\alpha = \frac{\sqrt{3}}{2}$$

The reference angle for α is $\frac{\pi}{3}$ but since $\sin\alpha$ is positive and $\cos\alpha$ negative, the angle x lies in II quadrant.

Thus $\quad \alpha = (\pi - \frac{\pi}{3}) + 2n\pi = \frac{2\pi}{3} + 2n\pi, n \in Z \qquad$ (iii)

Substituting (iii) in (ii) gives

$$\Rightarrow \quad -\sin\theta + \sqrt{3}\cos\theta = 2\sin(\theta + \alpha)$$

$$= 2\sin(\theta + \frac{2\pi}{3}) = 0$$

$$\Rightarrow \quad \theta + \alpha = k\pi \;,\; k \in Z$$

$$\Rightarrow \quad \theta = k\pi - \frac{2\pi}{3} - 2n\pi \;,\; n \in Z$$

$$\text{or} \quad \theta = -\frac{2\pi}{3} + m\pi \,,\, m \in Z$$

$$\text{S.S.} = \{ -\frac{2\pi}{3} + m\pi , m \in Z \} = \{ \frac{2\pi}{3} + m\pi , m \in Z \}$$

EXERCISE 12.6

1. Solve each equation giving general solutions.

(i) $\cos x = \frac{\sqrt{3}}{2}$ (ii) $\sin x = \frac{1}{2}$

(iii) $\tan x = -\sqrt{3}$ (iv) $\cos(2\theta - \frac{\pi}{2}) = -1$

(v) $\sec\frac{3\theta}{2} = -2$ (vi) $4\cos^2 x - 1 = 0$

2. Solve each equation. Use exact values in the given interval.

(i) $(\sin x)(\cos x) = 0$, $0 \le x \le 360°$

(ii) $(\sin x)(\cot x) = 0$, $0 \le x \le 2\pi$

(iii) $(\sec x - 2)(2\sin x - 1) = 0$, $0 \le x \le 2\pi$

(iv) $(\operatorname{cosec} x - 2)(2\cos x - 1) = 0$, $0 \le x \le 2\pi$

3. Find the solution sets of the following equations.

(i) $\cos\theta = \sin\theta$ (ii) $\tan\theta = 2\sin\theta$ (iii) $\sin\theta = \operatorname{cosec}\theta$

(iv) $4\cos^2(\frac{\theta}{2}) - 3 = 0$ (v) $\sin x \cos x = \frac{\sqrt{3}}{4}$ (vi) $\sin 2\theta + \sin\theta = 0$

4. Solve the following equations.

(i) $2\sin^2 x - 3\sin x + 1 = 0$ (ii) $\cos^2 x \sin x = 2$

(iii) $\cos^2 x - \sin^2 x = \sin x$ (iv) $\cos 2x + \cos x + 1 = 0$

(v) $1 - \sin x = 2\cos^2 x$ (vi) $\tan^2 x = \frac{3}{2}\sec x$

(vii) $3 - \sin x = \cos 2x$ (viii) $\sin\theta + \cos\theta = 1$

REVIEW EXERCISE 12

1. **Choose the correct option.**

(i) Solve $\sin 4x \cos x + \cos 4x \sin x = -1$ for all radian solutions.

(a) $\frac{\pi}{5}+\frac{2\pi}{5}k$ (b) $\frac{3\pi}{10}+\frac{2\pi}{5}k$ (c) $\frac{\pi}{2}+\frac{2\pi}{3}k$ (d) $\frac{\pi}{3}+\frac{2\pi}{3}k$

(ii) $\text{Tan}^{-1}\sqrt{3} - \text{Sec}^{-1}(-2)$ is equal to

(a) π (b) $-\pi/3$ (c) $\pi/3$ (d) $2\pi/3$

(iii) If $\text{Sin}^{-1}x = y$, then

(a) $0<y<\pi$ (b) $-\pi/2 \le y \le \pi/2$ (c) $0<y<\pi$ (d) $-\pi/2 < y < \pi/2$

(iv) $\sin(\text{Tan}^{-1}x)$, $|x| < 1$ is equal to

(a) $\frac{x}{\sqrt{1-x^2}}$ (b) $\frac{1}{\sqrt{1-x^2}}$ (c) $\frac{1}{\sqrt{1+x^2}}$ (d) $\frac{x}{\sqrt{1+x^2}}$

(v) $\text{Tan}^{-1}\left(\frac{x}{y}\right) - \text{Tan}^{-1}\left(\frac{x-y}{x+y}\right)$ is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$

2. Find the period of each function.

(i) $-2\operatorname{cosec} \pi x$ (ii) $6\tan \pi x$ (iii) $\frac{9}{5}\cos\left(-\frac{3\pi}{2}x\right)$

3. Solve the following equations.

(i) $\sin 2x = \cos x$ (ii) $\sin^2 x + \cos x = 1$ (iii) $\text{coec}x = \sqrt{3} + \cot x$

4. Prove the following.

(i) $2\,\text{Tan}^{-1}\frac{2}{3} = \text{Sin}^{-1}\frac{12}{13}$ (ii) $\text{Tan}^{-1}\frac{3}{4} + \text{Tan}^{-1}\frac{3}{5} - \text{Tan}^{-1}\frac{8}{19} = \frac{\pi}{4}$

(iii) $\text{Sin}^{-1}\frac{4}{5} + \text{Sin}^{-1}\frac{5}{13} - \text{Sin}^{-1}\frac{16}{65} = \frac{\pi}{2}$ (iv) $\text{Tan}^{-1}\frac{1}{11} + \text{Tan}^{-1}\frac{5}{6} = \text{Tan}^{-1}\frac{1}{3}\ \text{Tan}^{-1}\frac{1}{2}$

5. Prove the following.

(i) $\text{Cos}^{-1}A - \text{Cos}^{-1}B = \text{Cos}^{-1}(AB + \sqrt{A^2 + B^2})$

(ii) $\text{Tan}^{-1}A - \text{Tan}^{-1}B = \text{Tan}^{-1}(A - B)$

Answers

EXERCISE 1.1

1. (i) 0 (ii) i (iii) i (iv) $-i$

3. (i) $1+12i$ (ii) $\frac{3}{4}-i$ (iii) $(\sqrt{2}+1)+(\sqrt{2}+1)i$

4. (i) $(a-2)+bi$ (ii) $-6+0i$ (iii) $2\sqrt{3}-7\sqrt{7}\,i$

5. (i) $-117-i$ (ii) $6+6i$ (iii) $7+\sqrt{6}\,i$

6. (i) $\frac{1}{2}-\frac{1}{2}i$ (ii) $\frac{-8}{65}-\frac{1}{65}i$ (iii) $\frac{7}{58}+\frac{3}{58}i$ (iv) $1-6i$

7. (i) $\sqrt{34}$ (ii) $\sqrt{65}$ (iii) $\frac{8}{13}+\frac{1}{13}i$

8. (i) $\frac{6}{13}-\frac{24}{13}i$ (ii) $-\frac{22}{41}-\frac{7}{41}i$ (iii) $\frac{6}{25}+\frac{8}{25}i$

9. $\frac{63}{25}+\frac{16}{25}i$

10. $2-2i$

11. (i) -2 (ii) 0

EXERCISE 1.2

4 (i) $-5-2i, \frac{5}{29}-\frac{2}{29}i$ (ii) $(-7,\ 9), \left(\frac{7}{130}, \frac{9}{130}\right)$

7. (i) $\frac{4}{29}+\frac{19}{29}i$, *Real part* $x=\frac{4}{29}$, *Imaginary part* $y=\frac{19}{29}$

(ii) $-\frac{3}{2}-\frac{1}{2}i$, *Real part* $x=-\frac{3}{2}$, *Imaginary part* $y=-\frac{1}{2}$

(iii) $-\frac{1}{2}-\frac{1}{2}i$, *Real part* $x=-\frac{1}{2}$, *Imaginary part* $y=\frac{1}{2}$

iv. $\dfrac{4a^2-b^2}{(4a^2+b^2)^2}+\dfrac{4abi}{(4a^2+b^2)^2}$, Real part $=\dfrac{4a^2-b^2}{(4a^2+b^2)^2}$ *and imaginary part* $=\dfrac{-4ab}{(4a^2+b^2)^2}$

v. -1, Real part $x=-1$ *and imaginary part* $y=0$

vi. $\dfrac{533}{169}+\dfrac{308}{169}i$, Real part $=\dfrac{533}{169}$ *and imaginary part* $=\dfrac{308}{169}$

EXERCISE 1.3

1. i. $z=-2+9i,\ w=2-6i$ ii. $z=4-i,\ w=1-i$ iii. $z=1,\ w=3-2i$

2. i. $(z+2)(z-1+3i)(z-1-3i)$ ii. $(\sqrt{3}z+\sqrt{7}i)(\sqrt{3}z-\sqrt{7}i)$
   iii. $(z+2i)(z-2i)$ iv. $(z-2)(z+i)(z-i)$

4. Yes, it is a solution

5. i. $-\dfrac{1}{2}\pm\dfrac{1}{2}i$ ii. $\dfrac{1}{2}\pm i\dfrac{\sqrt{3}}{2}$ iii. $1\pm\sqrt{1-i}$ iv. $\pm 2i$

6. i. $\pm\dfrac{1}{2}\pm\dfrac{1}{2}i\sqrt{3}$ ii. $-2, 1\pm i\sqrt{3}$ iii. $0, \dfrac{3}{2}\pm\dfrac{1}{2}i\sqrt{3}$ iv. $1, \dfrac{-1}{2}\pm\dfrac{1}{2}i\sqrt{3}$

REVIEW EXERCISE 1

1. i. b ii. c iii. a iv. d v. c vi. b vii. b

3. i. $6+10i$ ii. $-4-4i$ iii. $-16+22i$ iv. $\dfrac{13}{37}+\dfrac{4}{37}i$

4. $\sqrt{2}$ 5. 2 6. $\dfrac{3}{25}+\dfrac{4}{25}i$ 7. $-i$ 8. $z=1\pm i$

EXERCISE 2.1

1. i. $[98]$ ii. $[-21\ \ 5\ \ -10]$ iii. $\begin{bmatrix}43\\44\end{bmatrix}$ iv. $\begin{bmatrix}6 & 16 & 26\\-8 & -18 & -28\end{bmatrix}$

2. $\begin{bmatrix} 7 & -20 & 1 \\ 6 & 1 & 11 \end{bmatrix}$ **5.** $a = -\frac{2}{3}, b = \frac{3}{2}$

6. (i) $X = \begin{bmatrix} 7 & 2 & 11 \\ 0 & 4 & 11 \end{bmatrix}$ (ii) $X = \begin{bmatrix} 3 & 5 & 3 \\ 3 & -3 & 3 \end{bmatrix}$

EXERCISE 2.2

1. $A_{11} = -4, A_{21} = 6, A_{23} = 6, A_{31} = -2, A_{32} = -1, A_{33} = 5, |A| = -14$

4. (i) -12 (ii) 92 (iii) 21 (iv) -11 **7.** (i) -2 (ii) 0

11. (i) Singular (ii) Non-singular (iii) Singular **12.** $\lambda = 0, \pm\sqrt{2}$

13. (i) $x = -1$ (ii) $x = 0, -1$ (iii) $x = 0, -9$

15. $\begin{bmatrix} \frac{15}{8} & \frac{-10}{8} & \frac{-2}{8} \\ \frac{7}{8} & \frac{-2}{8} & \frac{-2}{8} \\ \frac{-3}{8} & \frac{2}{8} & \frac{2}{8} \end{bmatrix}$

EXERCISE 2.3

1. (i) $\begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & -8 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{bmatrix}$ **2.** (i) $\frac{1}{49}\begin{bmatrix} 3 & 16 & -5 \\ -6 & 17 & 10 \\ 5 & -6 & 8 \end{bmatrix}$

(ii) $\begin{bmatrix} \frac{-17}{2} & \frac{31}{2} & -11 \\ \frac{-5}{2} & \frac{9}{2} & -3 \\ 4 & -7 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{-1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$ (iv) $\begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

3. (i) 3 (ii) 2 4. 2

EXERCISE 2.4

1. (i) $x = 3, \quad y = 1, \quad z = 2$ (ii) $x = 1, \quad y = \frac{2}{3}, \quad z = \frac{-2}{3}$

2. (i) $x = -1, \quad y = 3, \quad z = 2$ (ii) $x = 6, \quad y = -2, \quad z = 4$

3. (i) $x = -2, \quad y = 1, \quad z = 3$ (ii) $x = 2, \quad y = -2, \quad z = 3$

4. (i) Trivial solution (ii) $x_1 = -t, \quad x_2 = -t, \quad x_3 = t$

5. $\lambda = 1, \quad x_1 = \frac{1}{3}t, \quad x_2 = \frac{-2}{3}t, \quad x_3 = t$

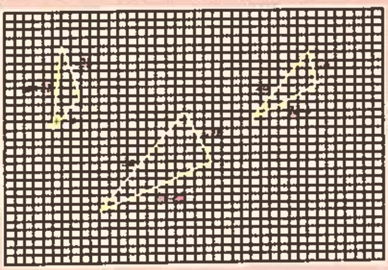
REVIEW EXERCISE 2

1. (i) c (ii) d (iii) c (iv) b (v) d (vi) c

2. $\begin{bmatrix} -193 \\ 232 \\ -78 \end{bmatrix}$ 4. -58 8. 51 9. $x=2, y=1, z=1$

EXERCISE 3.1

1. (i) $-a$ (ii) $-b$ (iii) $-c$ (iv) $2b$ (v) $2c$

2. 

3. (i) $\frac{1}{2}p$ (ii) $q-p$ (iii) $\frac{3}{4}q$ (iv) $\frac{3}{4}q-\frac{1}{2}p$

4. (i) $\frac{3}{2}b$ (ii) $a+\frac{3}{2}b$ (iii) $a+\frac{1}{2}b$

5. (i) $\frac{1}{2}a$ (ii) $b-\frac{1}{2}a$ (iii) $\frac{1}{3}b-\frac{1}{6}a$ (iv) $\frac{1}{3}a+\frac{1}{3}b$

6. (i) $\vec{r}-\vec{p}$ (ii) $\frac{1}{2}\vec{r}-\frac{1}{2}\vec{p}$ (iii) $\frac{1}{2}\vec{p}+\frac{1}{2}\vec{r}$

7. $\vec{x}+\vec{y}, \vec{y}, \vec{y}-\vec{x}, \vec{y}-2\vec{x}$

EXERCISE 3.2

1. (i) $-\vec{i}+\vec{j}$ (ii) $13\vec{i}-21\vec{j}$ (iii) $10\vec{i}-16\vec{j}$ (iv) $\sqrt{5}$ (v) $\sqrt{34}-\sqrt{13}$ (vi) $\sqrt{34}/\sqrt{13}$

2. (i) $\vec{i}$ (ii) $\frac{3}{5}\vec{i}-\frac{4}{5}\vec{j}$ (iii) $\frac{1}{\sqrt{6}}\vec{i}+\frac{1}{\sqrt{6}}\vec{j}-\frac{2}{\sqrt{6}}\vec{k}$ (iv) $\frac{\sqrt{3}}{2}\vec{i}-\frac{1}{2}\vec{j}$

3. $p=-4, q=1$

4. $-2\pm\sqrt{21}$

5. Length of $\overline{AB}=2\sqrt{29}$, unit vector in the direction of $\overline{AB}=\frac{5}{\sqrt{29}}i+\frac{2}{\sqrt{29}}j$

6. $-1, -4$

7. (i) components; 3, −3, magnitude $3\sqrt{2}$ (ii) components; 4, 2, magnitude $2\sqrt{5}$ (iii) components; 3, −2, 1, magnitude $\sqrt{14}$ (iv) components; −1, −6, −3, magnitude $\sqrt{46}$

8. (i) Q(−1, 1)

(ii) $P(-5,6)$ (iii) $Q(1,2,-5)$ (iv) $P(1,2,8)$ **9.** $2\vec{i}-4\vec{j}+4\vec{k}$

10. Internally $-\frac{1}{3}\vec{i}+\frac{4}{3}\vec{j}+\frac{1}{3}\vec{k}$, Externally $-3\vec{i}+3\vec{k}$ **11.** (i) $\frac{8}{7}\vec{i}+\frac{27}{7}\vec{j}$

(ii) $6\vec{i}+17\vec{j}$ **12.** No real value of α **13.** $z=-3$

EXERCISE 3.3

1. (i) -4 (ii) 15 (iii) 11 (iv) -12 (v) 29

2. $\frac{4}{13}\vec{i}+\frac{3}{13}\vec{j}-\frac{12}{13}\vec{k}$

3. (i) 90^0 (ii) 73^0 (*approximately*) (iii) 99^0 (*approximately*)

5. 4 **6.** (i) $m=\frac{26}{27}$ (ii) $m=\frac{27}{38}$

7. (i) $\frac{7}{15}, \frac{14}{17}$ (ii) $\frac{4}{\sqrt{3}}, \frac{12}{\sqrt{14}}$ **8.** $\frac{1}{2}$ **9.** work = 6 units **10.** 12 units

EXERCISE 3.4

1. (i) $3i$ (ii) $-3i-2j$ (iii) $-i+36j+22k$

3. (i) $\frac{1}{5\sqrt{3}}(-i+7j+5k)$ (ii) $\frac{1}{\sqrt{803}}(-25i+3j+13k)$

4. (i) $-19i-2j+9k$ (ii) $-3i+6j+3k$ (iii) $38i+4j-18k$

5. (i) 15 (ii) $\sqrt{19}$

6. (i) $-6\vec{i}+\vec{j}+4\vec{k}$ (ii) $22\vec{i}+3\vec{j}-12\vec{k}$ **8.** (i) $\frac{-5\vec{i}-\vec{j}+3\vec{k}}{\sqrt{35}}$

(ii) $10\frac{5\vec{i}+6\vec{j}+2\vec{k}}{\sqrt{65}}$ **9.** (i) $\sqrt{110}$ (ii) $\frac{\sqrt{321}}{2}$

EXERCISE 3.5

1. -10 2. 57 5. (i) 14 (ii) 9 (iii) 25

6. *Yes, they lie in a plane.* 7. (i) $\frac{53}{7}$ (ii) 1 (iii) $\frac{12}{5}$

8. (i) 3 (ii) $\frac{5}{6}$ 9. (i) 1 (ii) -1

REVIEW EXERCISE 3

1. (i) a (ii) a (iii) d (iv) b (v) b (vi) a (vii) a (viii) c

2. $\lambda = -9, \mu = 27$ 3. $\frac{1}{7}(6i - 3j + 2k)$ 4. 0 5. $\frac{8}{7}$

6. $-\frac{21}{2}$ 7. $\theta = \frac{\pi}{3}$ 8. $\frac{7}{2}$ *square units.* 9. $\sqrt{3336}$ *square units.*

EXCERCISE 4.1

1. (i) Finite (ii) Infinite (iii) Infinite (iv) Finite

2. (i) $1, 3, 6, 10, \ldots$ (ii) $4, -8, 16, -32$ (iii) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$ (iv) $0, 0, 1, 4, \ldots$

3. (i) $\frac{n}{n+1}$ (ii) $(-1)^{n+1} 2n$ (iii) $(-1)^{n+1}$ 4. (i) $3, 2, 3, 2, 3$ (ii) $3, 3, \frac{3}{2}, \frac{1}{2}, \frac{1}{8}$

5. (i) $-1+1+3+5+7+9$ (ii) $-1+2-4+8-16$ (iii) $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots$

(iv) $1+\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^3+\cdots$ 6. (i) $1, 5, 10, 10, 5, 1, 0, 0, 0, \ldots$

(ii) $1, 6, 15, 20, 15, 6, 1, 0, 0, 0, \cdots$ (iii) $1, 8, 28, 56, 70, 56, 28, 8, 1, 0, 0, 0$

EXCERCISE 4.2

1. 44 **2.** 38 **3.** n=26 **4.** 21 **5.** $\log(ab^{n-1})$

6. $k=\frac{7}{2}$, the sequence is 14, 19, 24, …… **7.** $\frac{5}{3}, 2, \frac{7}{3}, \ldots\ldots$

9. 3m **10.** 15135 **11.** 45 hours **12.** $ 18500 **13.** (i) 15

(ii) $\frac{7}{24}$ (iii) -111 (iv) a^2+b^2 **14.** (i) $14\frac{3}{4}, 23\frac{1}{2}, 32\frac{1}{4}$

(ii) 20, 23, 26, 29 **15.** 0 **16.** $\frac{11}{2}, 6, \frac{13}{2}, 7, \frac{15}{2}$ **17.** 8

EXCERCISE 4.3

1. (i) 20^{th} term: -29, sum: -200 (ii) 11^{th} term: $-\frac{1}{3}$, sum: $\frac{44}{3}$

2. (i) $a_{17}=50$; $S_{17}=$ [illegible] (ii) $a_{21}=60$; $d=5$; $n=21$

(iii) $n=9$; $a_9=57$ (iv) $n=15$; $d=\frac{2}{7}$; $a_1=0$ **3.** 12375

4. 8, 12, 16 ; 16, 12, 8 **5.** 2, 4, 6, 8 ; 8, 6, 4, 2 **6.** -21

7. $n(3n-4)$ **9.** 21978 **10.** Rs.280, Rs.260, Rs.240, Rs.220

11. 576 feet **12.** Rs.465 **13.** 3140 **14.** $\frac{66}{17}, \frac{115}{17}, \ldots\ldots, \frac{801}{17}$

EXERCISE 4.4

1. (i) 5, 15, 45, 135, 405 (ii) $8, -4, 2, -1, \frac{1}{2}$ (iii) $-\frac{9}{16}, \frac{3}{8}, -\frac{1}{4}, \frac{1}{6}, -\frac{1}{9}$

(iv) $\frac{x}{y}, -1, \frac{y}{x}, -\frac{y^2}{x^2}, \frac{y^3}{x^3}$ **2.** 3 ; $r=\pm 3$ **3.** $-2^{-\frac{3}{2}}$ **4.** 11

5. $x=13$; 20,10,5 **8.** (i) 2.92 or $-$ 2.92 (ii) 36 or -36 (iii) $\pm\sqrt{x^2-y^2}$

(iv) Does not exit **9.** (i) $\frac{16}{3}, 8, 12, 18, 27$ (ii) $-7, \frac{7}{2}, -\frac{7}{4}, \frac{7}{8}, -\frac{7}{16}, \frac{7}{32}$

10. 49,1 **12.** $n=-\frac{1}{2}$

EXERCISE 4.5

1. (i) $3(2^{10}-1)$ (ii) $\frac{255}{7}$ (iii) 2032 (iv) $\frac{64}{65}$ (v) $4+2\sqrt{2}$ (vi) $-\frac{463}{192}$

2. (i) $n=7, S_7=43$ (ii) $n=9, a_1=256, S_9=511$ (iii) $a_1=3, n=6$

3. $4, 2, 1, \frac{1}{2}, \frac{1}{4}; \quad S=8$ **4.** (i) $\frac{8}{9}$ (ii) $\frac{18}{11}$ (iii) $\frac{97}{45}$ (iv) $\frac{41}{333}$

5. 3 **7.** $1-\frac{1}{2^n}$ **8.** 18, 12, 8 or 8, 12, 18 **9.** 2 **10.** 7

12. $a_1=\frac{24}{5}, r=\frac{1}{5}; \frac{24}{5}+\frac{24}{5^2}+\ldots$ **14.** $29\frac{27}{32}$ ft

15. Rs. 16384; Rs. 1073741823 **16.** $\frac{3}{2}$

EXERCISE 4.6

1. (i) $\frac{1}{26}$ (ii) $\frac{2}{13}$ (iii) 1 **2.** $-\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7} - \frac{1}{9}, -\frac{1}{11}$ **3.** $-\frac{1}{18}$

4. (i) $A=2.925, H=2.91, G=\pm 2.92$ (ii) $A=-111, H=-11.68, G=\pm 36$

(iii) $A=x, G=\pm\sqrt{x^2-y^2}, H=\frac{x^2-y^2}{x}$ **5.** -1 **6.** 16, 24

7. $\frac{35}{23}, \frac{35}{31}, \frac{35}{39}, \frac{35}{47}$ **9.** 4 and 12 ; 12 and 4 **10.** 3 and 12 ; 12 and 3

REVIEW EXERCISE 4

1. (i) c (ii) b (iii) a (iv) b (v) d (vi) d (vii) b (viii) b

(ix) b (x) d (xi) c **2.** (i) $t_n = 9-n$

(ii) The progression is 8, 7, 6, 5,… (iii) $t_{10} = -1$ (iv) $S_n=\frac{n}{2}(17-n), S_{10}=35$

3. n = 18 or 19 **4.** 5,11,17,.... and nth term is 6n–1 **5.** 156375

6. $\frac{5}{3}$ **8.** 18 **9.** 2, 6 and 18 (or) 18, 6 and 2 **10.** 25 **11.** 196.875 feet

EXERCISE 5.1

1. (i) $\frac{n}{3}(4n^2-1)$ (ii) $\frac{n(n+1)^2(n+2)}{12}$ (iii) $\frac{2n(n+1)(2n+1)}{3}$

(iv) $n(2n^3-n)$ (v) $n(16n^3-16n^2-2n+3)$ **2.** $33\times100\times101$

3. 50×3333 **4.** $\frac{n(n+1)^2}{2}$ **5.** $\frac{n}{6}(2n^2+3n+7)$

6. $\frac{n(n+1)(2n+1)(n+3)}{4}$ **7.** $\frac{n(n+1)(n+8)(n+9)}{4}$ **8.** $\frac{n}{3}(32n^2+54n+25)$

9. (i) $\frac{n}{2}(n+1)(n^2+3n+1)$ (ii) $4^{n+1}-4-n(n+1)(n^2-n-1)$

EXERCISE 5.2

1. (i) $2+(n-1)2^{n+1}$ (ii) $\frac{1-(3n-2)x^n}{(1-x)}+\frac{3x(1-x^{n-1})}{(1-x)^2}$

(iii) $\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}$ (iv) $2+4(1-\frac{1}{2^{n-1}})-\frac{2n-1}{2^{n-1}}$ (v) $\frac{1-(6n-5)(-x)^n}{1+x}-\frac{6x[1-(-x)^{n-1}]}{(1+x)^2}$

2. (i) $\frac{1+7x}{(1-x)^2}+\frac{8x^2}{(1-x)^3}$ (ii) $\frac{9}{2}$ **3.** $n2^{n-1}$ **4.** 9 **5.** $\frac{1}{4}$

EXERCISE 5.3

1. $3n^2+1;\frac{1}{2}.n(2n^2+3n+3)$ **2.** $3n^2+n;n(n+1)^2$

3. $n^2+3n;\frac{1}{3}.n(n+1)(n+5)$ **4.** $3^{n-1}+2;\frac{1}{n}(3^n-1)+2n$

5. $3(2^n-1);3(2^{n+1}-n-2)$ **6.** $6^{n-1}+27;\frac{5^n-1}{4}+27n$

Answers

EXERCISE 5.4

1. (i) $\frac{n}{n+1}$ (ii) $\frac{n}{2n+1}$ (iii) $\frac{1}{6}$ (iv) $\frac{1}{36}$

2. $\frac{n}{2(3n+2)}$ 3. $\frac{n-1}{n}$ 4. $\frac{n}{4(n+4)}$

REVIEW EXERCISE 5

1. (i) b (ii) c (iii) c (iv) b (v) c (vi) c (vii) a (viii) a

2. $\frac{n(n+1)(n+2)}{3}$ 3. $\frac{1}{4}n(n+1)(n+4)(n+5)$ 4. $\frac{1}{24}$ 5. $\frac{(7n-2)x^n}{1-x}$

6. $\frac{n}{n+1}$ 7. (i) $\frac{n(n^3+4n^2+4n-1)}{2}$ (ii) $\frac{n(n+1)(3n^2+5n+1)}{6}$

8. (i) $\frac{n^2(n+1)^2}{4}+\frac{3}{2}(3^n-1)$ (ii) $\frac{n(n+1)(4n+11)}{6}$

(iii) $\frac{1}{12}n(n+1)[3n^2+23n+34]$ (iv) $\frac{n}{3}(4n^2-1)$ 9. (i) $\frac{n}{3}(n^2+3n+5)$

(ii) $\frac{n}{2}+\frac{3}{4}(3^n-1)$ 10. $2(1-\frac{1}{2^n})$, $2(n-1)+\frac{1}{2^{n-1}}$

EXERCISE 6.1

1. (i) 4200 (ii) $\frac{5}{16}$ (iii) $\frac{1}{(n+1)n}$ (iv) 252 2. (i) $\frac{19!}{13!}$

(ii) $2^6 \cdot 6!$ (iii) $\frac{(n+1)!}{(n-2)!}$ (iv) $\frac{(n+2)!}{3(n-1)!}$ 4. (i) 6 (ii) 9

EXERCISE 6.2

1. (i) 720 (ii) 380 (iii) 3,360 2. (i) 11 (ii) 9 (iii) 5

4. 40320 5. 5040 6. 120, number of even numbers is 48 7. (i) 125

(ii) 60 **8.** 2880 **9.** 1956 **10.** (i) 3360 (ii) 5040 **11.** 125

12. (i) 6720 (ii) 151200 (iii) 50400 (iv) 180 **13.** 37800 (i) 15120

(ii) 3360 (iii) 5040 (iv) 22680 (v) 7560 (vi) 30240 **14.** 12 **15.** 120

EXERCISE 6.3

1. (i) 9 (ii) 8 (iii) 4 , 5 **2.** 7 ; 4 **3.** 5 **5.** (i) 66 (ii) 220 **6.** 9

7. 190 **8.** 35 **9.** (i) 525 (ii) 1287 (iii) 1281 (iv) 231

EXERCISE 6.4

1. (i) $\frac{1}{6}$ (ii) 0 (iii) 1 (iv) $\frac{1}{3}$ (v) $\frac{1}{2}$ **2.** (i) $\frac{4}{91}$ (ii) $\frac{4}{555}$

3. (i) $\frac{1}{256}$ (ii) $\frac{1}{32}$ (iii) $\frac{7}{64}$ (iv) $\frac{37}{256}$ **4.** (i) $\frac{1}{8}$

(ii) $\frac{3}{8}$ (iii) $\frac{3}{8}$ (iv) $\frac{7}{8}$ (v) $\frac{1}{2}$ (vi) $\frac{1}{8}$ **5.** (i) $\frac{10}{21}$ (ii) $\frac{5}{21}$

6. (i) $\frac{1}{13}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{4}$ (iv) $\frac{3}{13}$ (v) $\frac{51}{52}$ **7.** (i) $\frac{1}{12}$ (ii) $\frac{5}{18}$

(iii) $\frac{5}{12}$ (iv) $\frac{1}{12}$ (v) $\frac{1}{6}$ (vi) $\frac{1}{9}$ (vii) $\frac{1}{2}$ (viii) $\frac{1}{2}$ (ix) $\frac{1}{3}$ (x) $\frac{5}{12}$

EXERCISE 6.5

1. $\frac{3}{10}$ **2.** (i) $\frac{1}{8}$ (ii) $\frac{1}{4}$ **3.** 0.1 **4.** $\frac{17}{30}$ **5.** $\frac{1}{9}$ **6.** $\frac{1}{3}$

7. $\frac{9}{13}$ **8.** $\frac{7}{9}$ **9.** (i) $\frac{1}{35}$ (ii) $\frac{2}{7}$ (iii) $\frac{24}{35}$ (iv) $\frac{11}{35}$ **10.** $\frac{316}{435}$

REVIEW EXERCISE 6

1. (i) a (ii) c (iii) a (iv) d (v) a (vi) d (vii) c (viii) b (ix) c

x. d **2.** i. $r = n - 1$ ii. ${}^{r}C_5 = 56$ **3.** $r = 41$ **4.** 27720

5. i. 72 ii. 24 **6.** $\frac{2}{5}$ **7.** i. 0.32 ii. 0.64 iii. 0.98

8. 1680 **9.** 360 **10.** $\frac{2}{n-1}$ **11.** $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{4}$

EXERCISE 7.2

1. i. $x^8 - \frac{4x^6}{y} + \frac{6x^4}{y^2} - \frac{4x^2}{y^3} + \frac{1}{y^4}$

ii. $1 + 7xy + 21x^2y^2 + 35x^3y^3 + 35x^4y^4 + 21x^5y^5 + 7x^6y^6 + x^7y^7$

iii. $\frac{1}{\sqrt{y}}\left[y^3 + 5y^2 + 10y + 10 + \frac{5}{y} + \frac{1}{y^2}\right]$

2. i. $560a^3$ ii. $-{}^{10}C_7\, 2^{-3}.3^7.x^3\, y^{-7}$ iii. $6x^3$ **3.** i. 2268 ii. 405

iii. $-{}^{21}C_7$ **4.** i. -1140 ii. $2^4.{}^{8}c_4$ iii. $-{}^{9}c_3 . \frac{2^6}{27}$

5. i. $70a^4b^4$ ii. $\frac{15309}{8}x^{13}$ *and* $\frac{5103}{16}x^{14}$ iii. $-252x^{10}y^5$

6. There is no constant term. **7.** i. 724 ii. $24\sqrt{2}$

iii. $2a^5 + 20a^3b^2 + 10ab^4$ **8.** $T_4 = -885735$ **9.** $T_6 = {}^{20}c_5\,(12)^{15}4^5$

EXERCISE 7.3

1. i. $1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5}{16}x^3 + \ldots$ ii. $1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \ldots$

iii. $4 + 4x - x^2 + \frac{2}{3}x^3 + \ldots$ **2.** i. 5.099 ii. 1.001 iii. 5.01330

3. $1 - x + \frac{1}{2}x^2 - \frac{1}{2}x^3$ **6.** $1 - \frac{3}{x} + \frac{9}{2x^2}$ **7.** 2, $\frac{3}{16}$ **9.** 4n

10. i. $\sqrt{\frac{2}{3}}$ ii. $\left(\frac{24}{7}\right)^{\frac{15}{7}}$

REVIEW EXERCISE 7

1. (i) a (ii) b (iii) a (iv) a (v) b (vi) c (vii) d (viii) a

2. $90720x^{12}y^4$ 3. -35840 4. $a = 4$ 5. 840 6. 0.951

EXCERCISE 8.1

1. (i) 1, −1, 5, 29 (ii) −3 or 2 (iii) x^2+3x+1 (iv) $2x+1+h$

2. (i) $f(6)=40,\ g(-1)=\frac{2}{3},\ h(4)=12,\ k\left(\frac{1}{2}\right)=\frac{5}{4}$ (ii) 7

3. (i) $\frac{1}{5}$ (ii) 3,5 (iii) $0, \pm 1$ (iv) $1, \pm\sqrt{5}$

4. (i) Domain f = R Range f = R

(ii) Domain f = R − (−4, 4) Range f = [0, ∞)

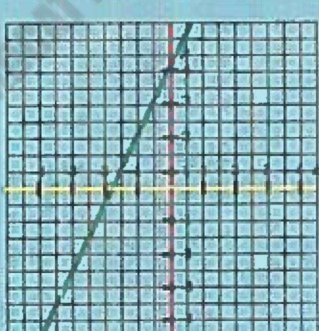
5. (i) $\frac{x+3}{2}$ (ii) $3x+15$ (iii) $2-5x$ (iv) $\frac{1}{2}(x-4)^2$

6. (i) $f^{-1}(x)=\sqrt[3]{x+2}$ (ii) $\sqrt[3]{5}$

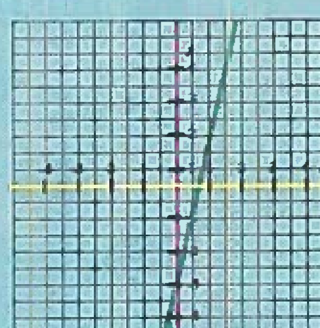
7. (i) $Dom(f)=R-\{3\}$, $Range(f)=R-\{1\}$ (ii) $Dom(f^{-1})=R-\{1\}$, $Range(f)=R-\{3\}$

EXERCISE 8.2

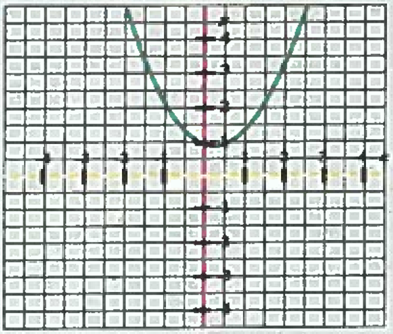
1. (i)

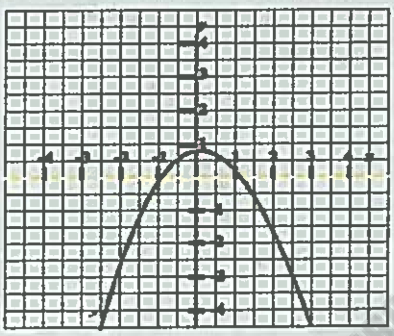


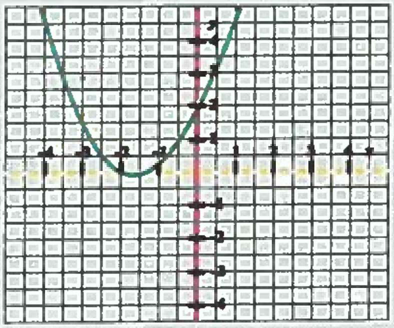
(ii)



2\.

i 

ii 

iii 

By changing the values of b in the quadratic function, the axis of symmetry of the graph moves in the x-direction

3\. i Vertex: (0, 0), y – intercept: 0, x – intercept : 0, Axis : 0, opens upward

ii Vertex: (0, 8), y – intercept: 8, x – intercepts: ± 2, Axis: 0, opens downward

iii Vertex: (3, 4), y – intercept: – 5, x – intercepts: 5 and 1, Axis: 3, opens downward

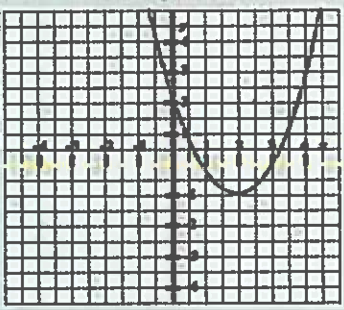
iv Vertex: $(-1, -\frac{7}{2})$, y – intercept: –3, x – intercept: 1, Axis: 1, –3, opens upward

4\. i ----(e) ii ----(c) iii ----(b) iv ----(a) v ----(f) vi ----(d)

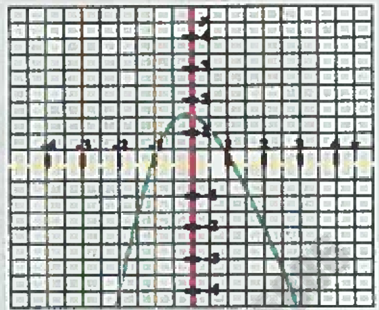
EXERCISE 8.3

1.

(i)

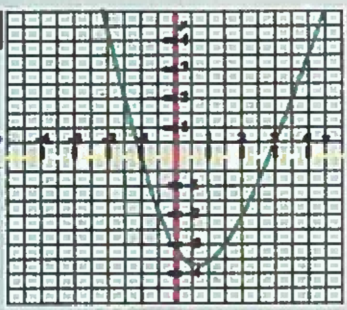


(ii)

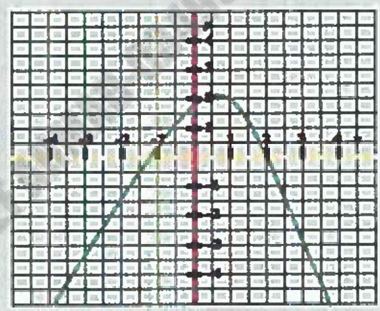


2.

(i)



(ii)

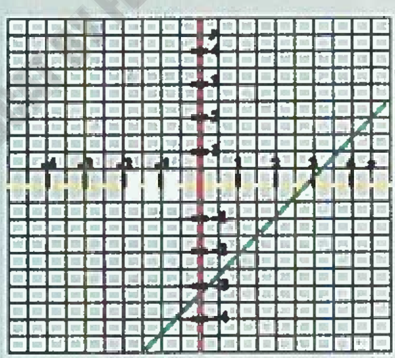


3. $y = x^2 - 7x + 12$

4. (i) $y = -\frac{1}{2}x^2 - x + \frac{15}{2}$ (ii) $y = -\frac{1}{6}x^2 + \frac{1}{2}x + \frac{35}{3}$

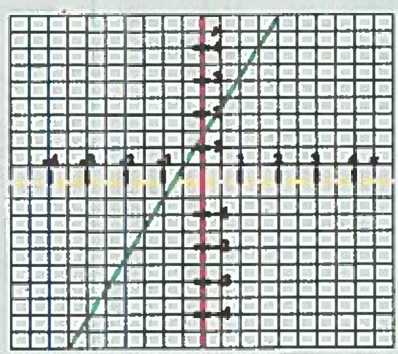
5.

(i)



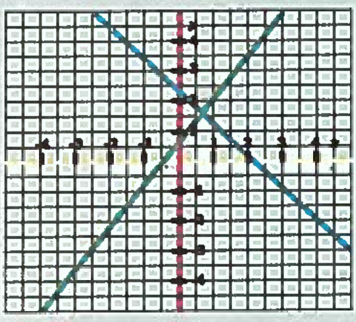
(3, 0), (0, 3)

(ii)



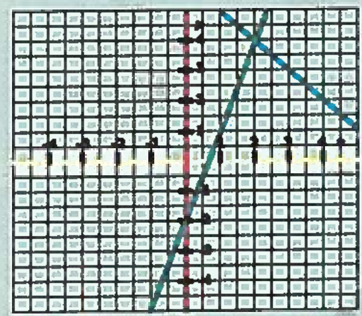
(3, 0), (0, 3)

6. i



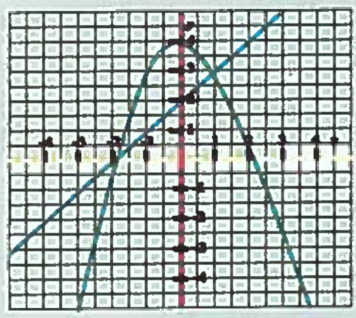
$\left(\frac{1}{3}, \frac{5}{3}\right)$

ii



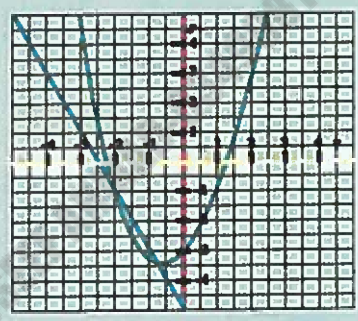
(2, 4)

7. i



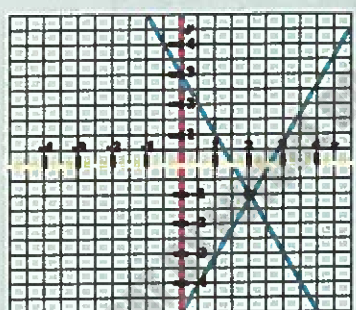
(1, 3), (−2, 0)

ii



(−1, 3), (−2, 1)

8.



(2, −2)

9. Air speed = 5 km/min, velocity of the wind = 1 km/ min.

REVIEW EXERCISE 8

1. i d ii b iii c iv b v a vi b vii d viii c

2. Domain $f = \left[-2\sqrt{3}, -\sqrt{3}\right] \cup \left[\sqrt{3}, 2\sqrt{3}\right]$

3. $f(x) = 3x^2 - 2x + 5$

4. i. $[2, \infty)$ ii. R iii. 14 iv. 10

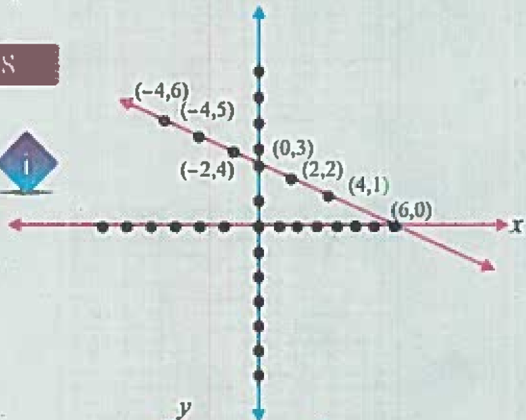
5. i. 32 ii. 82.4

6. i. 2 ii. $\frac{4}{3}$

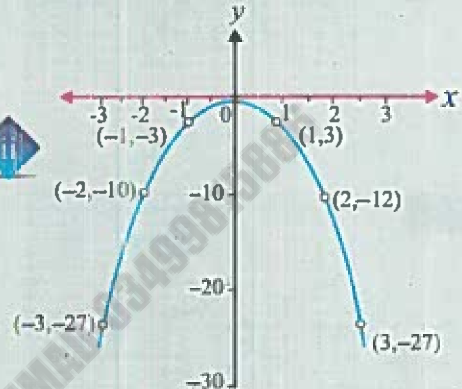
7. $a = 2, b = -2y$

8.

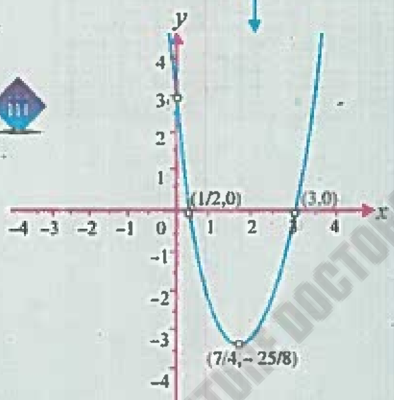
i.



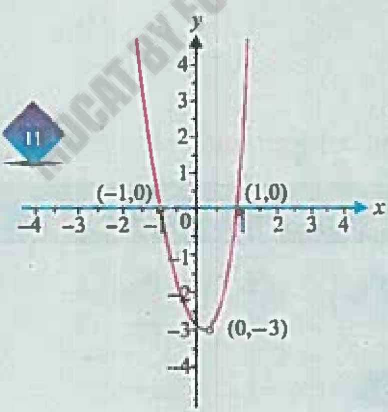
ii.



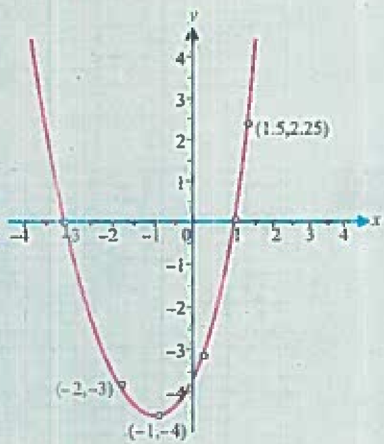
iii.



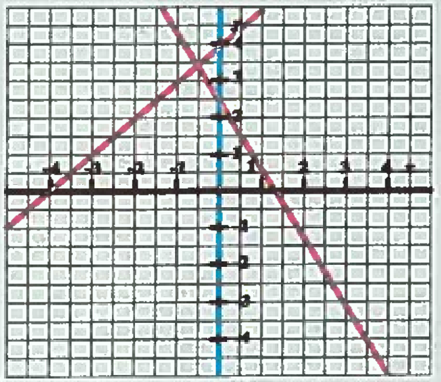
ii.



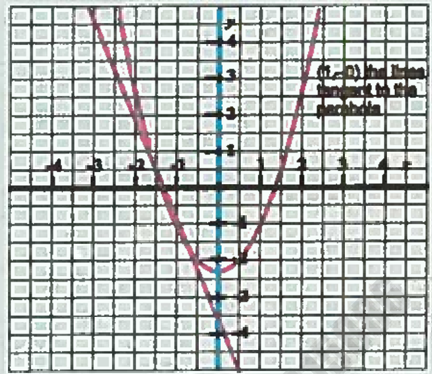
9. i.



10.

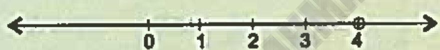
i 

$\left(-\frac{1}{3}, \frac{11}{3}\right)$

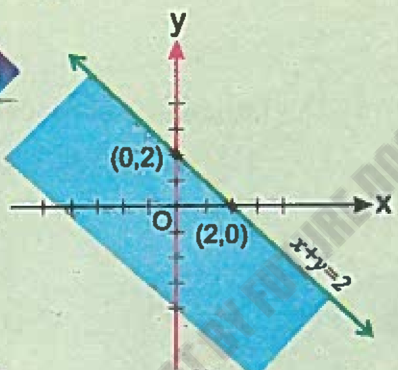
ii 

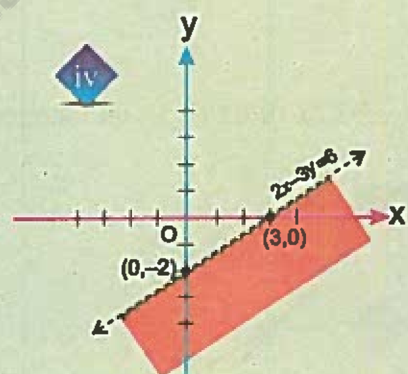
EXERCISE 9.1

1. i $x < 4$

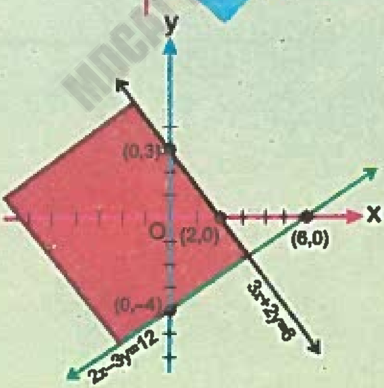


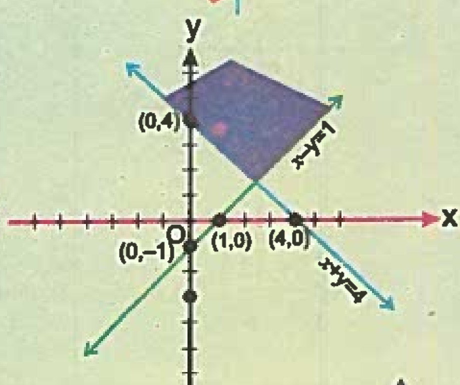
ii $x \geq -2$

iii 

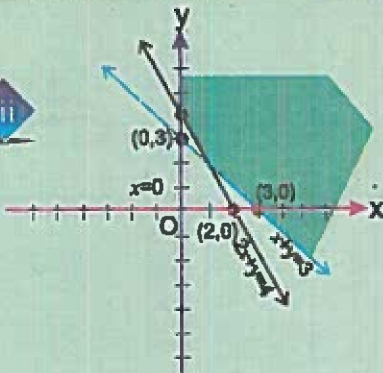
iv 

2.

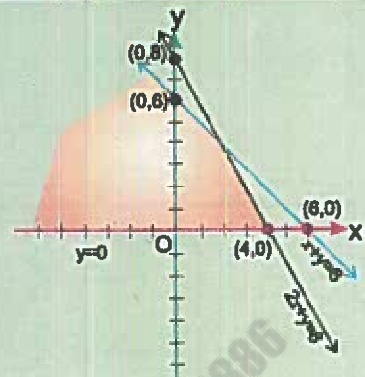
i 

ii 

iii

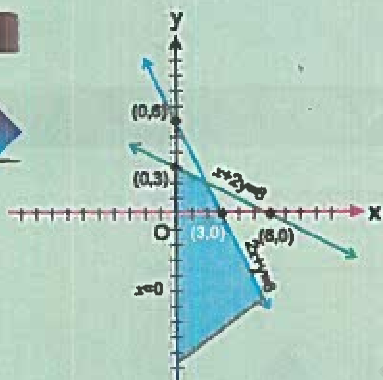


iv



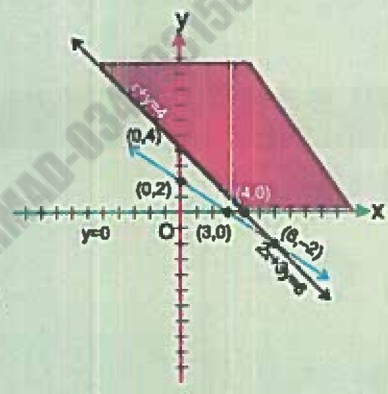
3.

i



(2,2), (0,6), (0,3); unbounded

ii



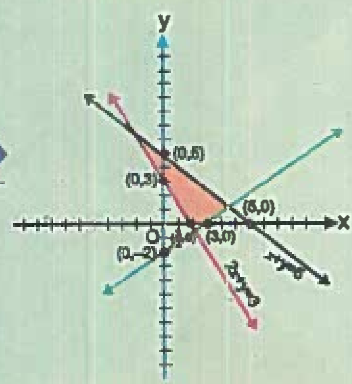
(6,–2), (3,0), (4,0); unbounded

4.

i

$\left(\frac{24}{7}, \frac{12}{7}\right)$, (–6,8), (5, – 3); bounded

ii

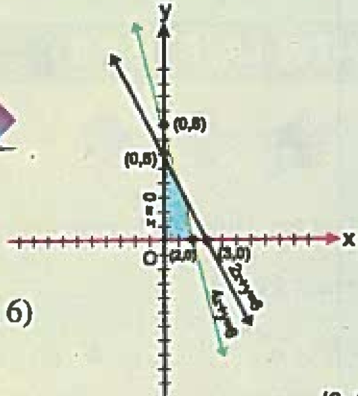


(–2, 7), $\left(\frac{5}{3}, \frac{1}{3}\right)$, $\left(\frac{7}{2}, \frac{3}{2}\right)$; bounded

EXERCISE 9.2

1.

(i)

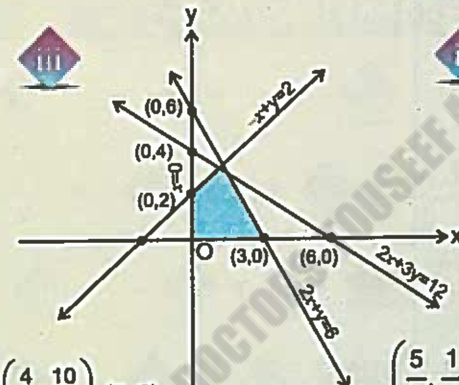


(0, 0), (2, 0), (1, 4), 0, 6)

(ii)

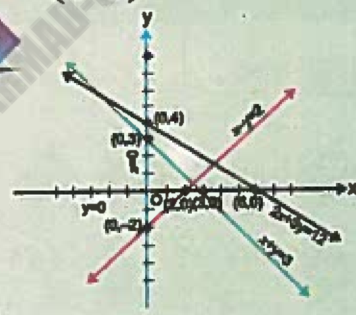
(0, 0), (5,0) $\left(\frac{1}{4},\frac{19}{4}\right)$, (0, 4) ; bounded

(iii)



(0, 0), (3,0), $\left(\frac{3}{2}, 3\right)$, $\left(\frac{4}{3},\frac{10}{3}\right)$, (0, 2)

(iv)



$\left(\frac{5}{2},\frac{1}{2}\right)$, $\left(\frac{18}{5},\frac{8}{5}\right)$, (0, 4), (0, 3)

2. (i) Maximum value is 12 at the corner point (6, 0).

(ii) Maximum value is 20 at the corner point (0, 4).

3. (i) Maximum value is 50 at the corner point (10, 0).
Minimum value is 4 at the corner point (0,2).

(ii) Maximum value is 84 at the corner point (0, 4).
Minimum value is 7 at the corner point (1, 0).

4. Maximum profit of Rs. 1140 if 16 bicycles of model A and 10 bicycles of model B are produced.

5. Maximum profit of Rs. 14000 if 200 units of product A and 400 units of product B are produced.

6. Maximum profit of Rs. 1760 if 8 lamps of model L_1 and 24 lamps of model L_2 are produced

REVIEW EXERCISE 9

1. i a ii c iii c iv c v d vi a

2. Maximum value of z is 24 at two different corner points $\left(\frac{24}{7}, \frac{24}{7}\right)$ and $(5, \frac{4}{3})$

3. Rs: 112, when $x = 2$ kg, $y = 4$ kg

4. Maximum value of z is 600 at A (120,0) and R (60,30)

EXERCISE 10.1

1. i $\sin 59^0$ ii $\cos 30^0$ iii $\cos 24^0$ iv $\sin 25^0$ v $\tan 52^0$ vi $\tan 23^0$

2. i $\frac{\sqrt{6}-\sqrt{2}}{4}$ ii $2+\sqrt{3}$ iii $-2-\sqrt{3}$ iv $2+\sqrt{3}$ v $\frac{\sqrt{6}+\sqrt{2}}{4}$ vi $\frac{\sqrt{6}+\sqrt{2}}{4}$

3. i 0 ii $-\frac{7}{24}$ iii $-\frac{7}{25}$ iv $\frac{24}{25}$

4. i $\frac{63}{65}$ ii $\frac{56}{65}$ iii $\frac{33}{56}$

5. i $\frac{33}{65}$ ii $\frac{-56}{65}$ iii $\frac{-33}{56}$

13. i $\gamma \sin(\theta + \phi)$ where $\sin \phi = \frac{3}{5}$, $\cos \phi = \frac{4}{5}$ and $r = 5$

ii $\gamma \sin(\theta + \phi)$ where $\sin \phi = \frac{8}{17}$, $\cos \phi = \frac{15}{17}$ and $r = 17$

iii $\gamma \sin(\theta + \phi)$ where $\sin \phi = \frac{-5}{\sqrt{29}}$, $\cos \phi = \frac{2}{\sqrt{29}}$ and $r = \sqrt{29}$

iv $\gamma \sin(\theta + \phi)$ where $\sin \phi = \frac{1}{\sqrt{2}}$, $\cos \phi = \frac{1}{\sqrt{2}}$ and $\gamma = \sqrt{2}$

EXERCISE 10.2

1. $-\frac{5}{13}, \frac{12}{13}, -\frac{5}{12}$

2. i $\frac{-120}{169}$ ii $\frac{119}{169}$ iii $\frac{-120}{119}$

3. i $\frac{-24}{25}$

(ii) $\frac{1}{\sqrt{5}}$ **4.** $\frac{\sqrt{5}}{\sqrt{7}}$ **5.** (i) $\frac{\sqrt{3}}{2}$ (ii) $\frac{-1}{2}$ **6.** (i) $\frac{\sqrt{2+\sqrt{3}}}{2}$

(ii) $\sqrt{3+2\sqrt{2}}$ (iii) $\frac{\sqrt{2+\sqrt{2}}}{2}$ (iv) $\frac{\sqrt{2+\sqrt{2}}}{2}$ (v) $\sqrt{7+4\sqrt{3}}$ (vi) $\frac{\sqrt{2+\sqrt{3}}}{2}$

8. $\frac{3}{8}+\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta$

EXERCISE 10.3

1. (i) $\cos 5x-\cos 7x$ (ii) $\frac{1}{2}\left[\sin 178^0-\sin 66^0\right]$ (iii) $\frac{1}{2}$ (sin A + sinB)

(iv) $\frac{1}{2}$ (cos P + cos Q) **2.** (i) $2\sin 40^0 \cos 3^0$ (ii) $-2\sin 59^0 \sin 23^0$

(iii) $2\cos\frac{P}{2}\sin\frac{Q}{2}$ (iv) $2\cos\frac{A}{2}\cos\frac{B}{2}$

REVIEW EXERCISE 10

1. (i) b (ii) c (iii) b (iv) d (v) d (vi) d (vii) a (viii) c

EXERCISE 11.1

1. (i) a=3, $b=3\sqrt{3}, \beta=60^0$ (ii) a=52.7, c=136.6, $\alpha=22.7^0$

(iii) $a=5\sqrt{2}, b=5\sqrt{2}, \alpha=45^0$ **2.** (i) $\alpha=62°$, b = 7.44, c = 15.86

(ii) $\alpha=68.5°$, a = 22.59, c = 24.28 (iii) $\alpha=88.22°$, $\beta=1.78°$, a = 449.78

3. 24.89m **4.** 52.9° **5.** 36.3m **6.** 45.3m **7.** 11.43m

8. 189.3m **9.** 61.4 feet **10.** 7.265cm

EXERCISE 11.2

1. (i) $\alpha=60°, \beta=30°, \gamma=90°$ (ii) $\alpha=25°, \beta=123°$, c = 152

(iii) a = 408, b = 166, $\beta=23.6°$ (iv) $\alpha=23°, \gamma=45°$, b = 57.6

v. $a = 3.83$, $\beta = 24.3°$, $\gamma = 55.3°$ vi. $\alpha = 106°20'$, $b = 159$, $c = 140$

vii. $a = 68$, $b = 112$, $\gamma = 75°$ viii. No triangle possible

ix. $a = 15.31$, $\beta = 30°26'$, $\gamma = 111°14'$

x. $b = 409.00$, $\alpha = 22°39'$, $\gamma = 46°59'$

2. i. $\alpha = 96°37'$ ii. $\beta = 80°0'38''$ iii. $\gamma = 87°55'$

3. i. $\alpha = 95.7°$, $\beta = 50.7°$, $\gamma = 33.6°$ ii. $\alpha = 4.0°$, $\beta = 31.6°$, $\gamma = 144.4°$ iii. $\alpha = 26.4°$, $\beta = 36.4°$, $\gamma = 117.2°$

4. 7.9cm, 14.8cm 5. 1879km apart

6. $\alpha = 43°17'$, $\beta = 64°26'$, $\gamma = 72°17'$ 7. 72.9cm

EXERCISE 11.3

The answers are in square units.

1. i. 369.42 ii. 83 iii. 680 iv. 564.7 v. 6.4 vi. 35.5 vii. 76662 viii. 400.5 ix. 651.7 x. 614.5 xi. 134736.6 xii. 2730.7

2. $c = 22.24$, $\gamma = 82°8'$ 3. Rs.1125 4. 787 ft^2

EXERCISE 11.4

1. i. $R = 3.0$, $r = 1.07$ ii. $R = 14.5$, $r = 6$ 2. $r = 8.16m$, Area $= 209\ m^2$

3. i. 33.07dm ii. 14.17dm 7. $3\sqrt{3}, 7\sqrt{3}, 8\sqrt{3}$

REVIEW EXERCISE 11

1. i. c ii. b iii. a iv. d v. d vi. c vii. c viii. b

2. i. $b = 1.42$, $\alpha = 17.2°$, $\gamma = 21.3°$ ii. No triangle possible.

iii $\alpha = 59°43'$, $\beta = 85°7'$, c = 10.4 iv c = 40.68, $\alpha = 81°43'$, $\beta = 41°17'$

v a = 14.74, $\beta = 70°39'$, $\gamma = 85°56'$ vi $\alpha = 42.8°$, b = 52, c = 84.7

vii $\gamma = 77.5°$, a = 7.05, b = 13.3 **3.** i 42.7° ii 43.7° **4.** i 82.5

ii 83.03° **5.** 57.1cm, 20.84 cm **6.** $25\sqrt{3}$m **7.** 168.93 ft **8.** 57.8 m

EXERCISE 12.1

1. i R, $[-3,3], \frac{2\pi}{3}$ ii $R-\{x \mid x=(2n+1)\pi ; n \in Z\}, R, 2\pi$

iii $R-\left\{x \mid x=n\frac{\pi}{2}; n \in Z\right\}, R-(-1,1), \pi$ iv $R, \{y \mid -1 \le y \le 1,\ y \in R\}, \frac{\pi}{2}$

v $R-\left\{x \mid x=(2n+1)\frac{\pi}{4};\ n \in Z\right\}, R-\{y \mid -6 \le y \le 6,\ y \in R\}, \pi$

vi $R-\left\{x \mid x=\frac{3n}{2};\ n \in Z\right\}, R, \frac{3}{2}$ vii $R-\left\{x \mid x=(2n+1)\frac{\pi}{2};\ n \in Z\right\}, R, \pi$

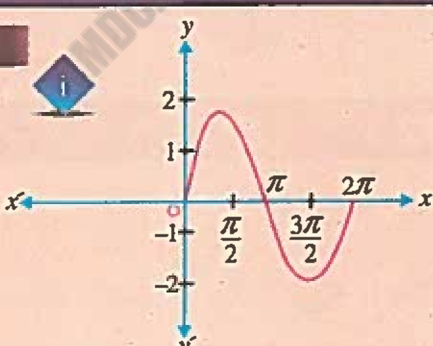
viii $R-\{x \mid x=n\pi;\ n \in Z\}, R-\left(-\frac{1}{2}, \frac{1}{2}\right), 2\pi$

ix $R-\{x \mid x=4n+2,\ n \in Z\}, R-(-1,1), 8$ **2.** i $-\frac{3}{2}$, $-\frac{5}{2}$

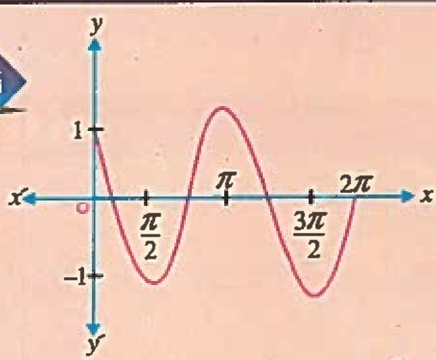
ii 9, 1 iii $\frac{1}{9}$, $\frac{1}{29}$ iv $\frac{1}{4}$, $-\frac{1}{4}$

EXERCISE 12.2

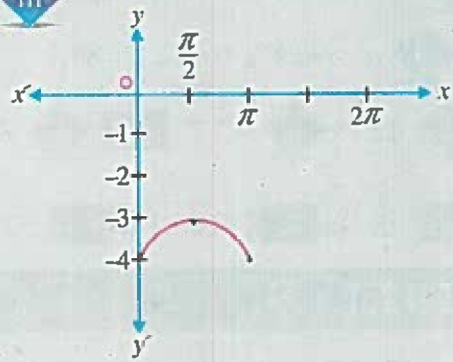
1. i



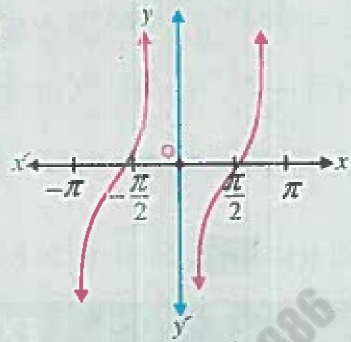
ii



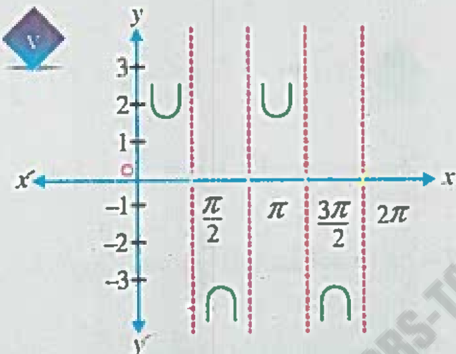
iii



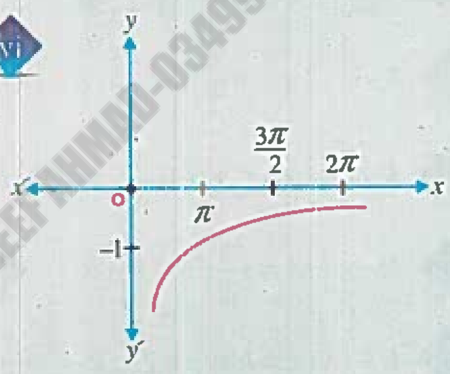
iv



v



vi



2. i Period π, Frequency $\frac{1}{\pi}$, Amplitude 1

ii Period $\frac{\pi}{3}$, Frequency $\frac{3}{\pi}$, Amplitude 1

iii Period 2, Frequency $\frac{1}{2}$, Amplitude 1

iv Period 4, Frequency $\frac{1}{4}$, Amplitude 1

EXERCISE 12.3

1. i $\theta=\frac{\pi}{4}+2n\pi$ or $\theta=\frac{3\pi}{4}+2n\pi; n\in Z$

ii $\theta=\frac{5\pi}{6}+2n\pi$ or $\theta=\frac{7\pi}{6}+2n\pi; n\in Z$ iii $\theta=\frac{\pi}{3}+n\pi; n\in Z$

iv $\theta=\frac{\pi}{3}+2n\pi; n\in Z$ v $\theta=\frac{3\pi}{4}+\pi n;\ n\in Z$

vi $\theta=\frac{7\pi}{6}+2n\pi$ or $\theta=\frac{11\pi}{6}+2n\pi,\ n\in Z$

EXERCISE 12.4

1. i $\left\{-\frac{\pi}{2}+2n\pi, n\in Z\right\}\cup\left\{\frac{3\pi}{2}-2n\pi, n\in Z\right\}$

ii $\left\{\frac{3\pi}{4}+2m, m\in Z\right\}\cup\left\{\frac{5\pi}{4}+2m\pi, m\in Z\right\}$

iii $\left\{-\frac{\pi}{6}+2n\pi, n\in Z\right\}\cup\left\{\frac{5\pi}{6}+n\pi, n\in Z\right\}$

2. i $\pi+2n\pi,\ n\in Z$ ii $\frac{1}{2}$ iii $\left\{-\frac{1}{2},\frac{1}{2}\right\}$ iv $\frac{-3}{4}$

3. i $\frac{\sqrt{2}}{2}$ ii $\frac{\sqrt{3}}{3}$ iii 2 iv $\sqrt{2}$ v $\frac{-\sqrt{2}}{2}$ vi $2\frac{\sqrt{3}}{3}$

4. i u ii u iii u iv $\sqrt{1-u^2}$

EXERCISE 12.5

1. i $\frac{\pi}{3}$ ii $\frac{\sqrt{3}}{2}$ 3. i $\frac{4}{5}$ ii Does not exist.

5. i $\text{Sin}^{-1}x=\text{Tan}^{-1}\frac{x}{\sqrt{1-x^2}}$, $-1<x<1$

ii $\text{Cos}^{-1}x=\text{Tan}^{-1}\frac{\sqrt{1-x^2}}{x}$, $0<x<1$

iii $\text{Cot}^{-1}x=\text{Tan}^{-1}\left(\frac{1}{x}\right)$, $0<x<\infty$ 7. $\text{Tan}^{-1}\frac{5}{6}$

EXERCISE 12.6

1. i $\left\{\frac{\pi}{6}+2k\pi\right\}\cup\left\{\frac{11\pi}{6}+2k\pi\right\}, k\in Z$

(ii) $\left\{\frac{\pi}{6}+2k\pi\right\}\cup\left\{\frac{5\pi}{6}+k\pi\right\}$, k is any integer. (iii) $\left\{\frac{2\pi}{3}+k\pi\right\}, k\in Z$

(iv) $\theta=\frac{3\pi}{4}+k\pi$, k any integer.

(v) $\theta\in\left\{\frac{4\pi}{9}+\frac{4}{3}k\pi, k\in Z\right\}\cup\left\{\frac{8\pi}{9}+\frac{4}{3}k\pi, k\in Z\right\}$

(vi) $\left\{\frac{\pi}{3}+2n\pi, n\in Z\right\}\cup\left\{\frac{5\pi}{3}+2n\pi, n\in Z\right\}\cup\left\{\frac{2\pi}{3}+2n\pi, n\in Z\right\}\cup\left\{\frac{4\pi}{3}+2n\pi, n\in Z\right\}$

2. (i) $0^\circ, 90^\circ, 180^\circ, 270^\circ$ (ii) $\frac{\pi}{2}, \frac{3\pi}{2}$ (iii) $\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}$

(iv) $\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{5\pi}{6}$ **3.** (i) $\left\{\frac{\pi}{4}+n\pi\right\}\cup\left\{\frac{5\pi}{4}+n\pi\right\}$

(ii) $\{k\pi\}\cup\left\{\frac{5\pi}{3}+2k\pi\right\}\cup\left\{\frac{\pi}{3}+2k\pi\right\}, k\in Z$

(iii) $\left\{\frac{\pi}{2}+2k\pi\right\}\cup\left\{\frac{3\pi}{2}+2k\pi\right\}$, k any integer.

(iv) $\left\{\frac{\pi}{3}+4k\pi\right\}\cup\left\{\frac{11\pi}{3}+4k\pi\right\}\cup\left\{\frac{5\pi}{3}+4k\pi\right\}\cup\left\{\frac{7\pi}{3}+4k\pi\right\}, k\in Z$

(v) $\left\{\frac{\pi}{6}+n\pi\right\}\cup\left\{\frac{\pi}{3}+n\pi\right\}$, $n\in Z$

(vi) $\{m\pi\}\cup\left\{\frac{2\pi}{3}+2m\pi\right\}\cup\left\{\frac{4\pi}{3}+2m\pi\right\}$, m is any integer.

4. (i) $\left\{\frac{\pi}{6}+2k\pi\right\}\cup\left\{\frac{5\pi}{6}+2k\pi\right\}\cup\left\{\frac{\pi}{2}+2k\pi\right\}$, k is any integer.

(ii) No real solution.

(iii) $\left\{\frac{\pi}{6}+2n\pi\right\}\cup\left\{\frac{5\pi}{6}+2n\pi\right\}\cup\left\{\frac{3\pi}{2}+2n\pi\right\}$, n is any integer.

iv $\left\{\frac{\pi}{2}+2n\pi\right\} \cup \left\{\frac{2\pi}{3}+2n\pi\right\} \cup \left\{\frac{4\pi}{3}+2n\pi\right\}$, n is any integer.

v $\left\{\frac{\pi}{2}+2m\pi\right\} \cup \left\{\frac{7\pi}{6}+2m\pi\right\} \cup \left\{\frac{11\pi}{6}+2m\pi\right\}$, m is any integer.

vi $\left\{\frac{\pi}{3}+2k\pi\right\} \cup \left\{\frac{5\pi}{3}+2k\pi\right\}$, k is any integer.

vii No real solution. viii $\left\{\frac{\pi}{2}+2k\pi\right\} \cup \{2k\pi\}$, k is any integer.

REVIEW EXERCISE 12

1. i b ii d iii b iv d v c 2. i 2 ii 1 iii $-\frac{4}{3}$

3. i $\left\{\frac{\pi}{2}+2k\pi\right\} \cup \left\{\frac{3\pi}{2}+2k\pi\right\} \cup \left\{\frac{\pi}{6}+2k\pi\right\} \cup \left\{\frac{5\pi}{6}+2k\pi\right\}, k \in Z$

ii $\left\{2k\pi+\frac{\pi}{2}\right\} \cup \left\{2k\pi+\frac{3\pi}{2}\right\} \cup \{2k\pi\}, k \in Z$ iii $\left\{2k\pi+\frac{2\pi}{3}\right\}, k \in Z$

The Authors

Prof. Dr. Gulzar Ali Khan (Retired)

Gulzar Ali Khan received his B.Sc. from Government Postgraduate College, Bannu in 1975, and M.Sc. from Gomal University, Dera Ismail Khan and Leeds University, U.K. in 1977 and 1981 respectively, and his Ph.D. from Birmingham University, U.K. in 1986 all in mathematics.

Dr. Khan taught at Gomal University from 1986 to 2000. He joined the University of Peshawar in 2001, He remained Chairman of the Department of Mathematics. He has published many research papers in his field of interest. Dr. Khan has been a member of the Advisory Committee, Ministry of Education, Government of Pakistan, Islamabad in the subject of Mathematics. He has also been a member of the National Review Committee (NRC) on mathematics. Dr. Khan is now retired.

Prof. Dr. Islam Noor (Retired)

Dr. Islam Noor is a retired Professor of Mathematics of the University of Peshawar. He obtained M.Sc. degree from the same University in 1972. He received M.S & Ph.D degrees from *Temple University (U.S.A)* in 1982 & 1984 respectively.

Dr. Noor has published numerous papers in reputed national & internal journals. He was a member of HEC curriculum committee for revising B.Sc, 4 years B.S and M.Sc. programs. He remained Chairman of the Department of Mathematics twice. He has organized seminars at Bara Gali campus, University of Peshawar for college teachers to cope with the changes made in the syllabus of F.Sc. & B.Sc.

Prof. Dr. Muhammad Shah

Dr. Muhammad Shah did his M.Sc from University of Peshawar in 1998. He did his Ph.D in 2012 from Quaid-i-Azam University Islamabad in computational Algebra. In 2010, he was awarded IRSIP scholarship of HEC for Birmingham University, UK.

Dr. Shah has been teaching Mathematics at postgraduate level since 2002 in different colleges of Khyber Pakhtunkhwa. He has been the Subject Specialist of Mathematics and Computer Science of Khyber Pakhtunkhwa Textbook Board since 2014 to 2017. He is the editor and reviewer of several textbooks of Mathematics and Computer Science.

مومن تو آپس میں بھائی بھائی ہیں۔
پس اپنے دو بھائیوں میں صلح کرا دیا کرو
اور اللہ سے ڈرتے رہو تا کہ تم پر رحم
کیا جائے۔

(سورۃ الحجرات: ۱۰)